Spring 1996 Qualifying Exam, Selected Solutions

DAY I

Question 1 Since $\sin(\pi z)$ is holomorphic we have that

$$f(z) := \frac{\pi z}{\sin(\pi z)}$$

is meromorphic. The singularities of f can only occur at integers; indeed if $sin(\pi z) = 0$ then

$$\frac{e^{-i\pi z}}{2i}(e^{i2\pi z} - 1) = 0$$

Since only the rightmost factor can vanish we have

$$2\pi i z = 2\pi i k$$

when and only when k is an integer. If $z = 0$ then

$$\lim_{z\to 0} f(z) = \lim_{z\to 0} \frac{\pi}{\pi\cos(\pi z)} = 1$$

by L'Hôpital's Rule and so f has a removable singularity there; for $k \neq 0$ integral, on the other hand,

$$\begin{aligned}\lim_{z\to k}(z-k)f(z) &= \lim_{z\to k} \frac{\pi z(z-k)}{\sin(\pi z)} \\ &= \Big(\lim_{z\to k} \frac{z-k}{\sin(\pi z)}\Big)(\pi k)\end{aligned}$$

By the continuity of the reciprocal where $z \neq 0$ and the fact that

$$(\sin(\pi z))'|_{z=k} = (-1)^k \pi \neq 0$$

we have that the residue of $f(z)$ at $z = k$ is:

$$\lim_{z\to k}(z-k)f(z) = \frac{(-1)^k}{\pi}(\pi k) = (-1)^k k$$

But $(-1)^k = (-1)^{-k}$, and so the residues at k and $-k$ cancel one another out. By the Residue Theorem, then,

$$\int_{|z|=100.5} f(z)\, dz = 0$$

Question 6 Bijectivity of $f : E \to E$[1] is equivalent to the fact that, for all $w \in E$, $f(z)-w$ has one and only one zero. Since $|f(z)| = 1$ on δE by hypothesis and $|w| < 1$, we have by Rouché's Theorem that $f(z)$ and $f(z) - w$ have the same number of zeros in E – i.e., exactly one.

[1] I assume, for purposes of this question, that the author intended E to be the unit disc.

DAY II

Question 1 We show that the following limit exists:

$$\lim_{h\to 0}\frac{F(\xi+h)-F(\xi)}{h} = \lim_{h\to 0}\int_{\mathbb{R}} f(x)e^{ix\xi}\frac{e^{ixh}-1}{h}\,dx$$

It will suffice to show that the limit and integral can be interchanged, for then

$$\begin{aligned}\lim_{h\to 0}\int_{\mathbb{R}} f(x)e^{ix\xi}\frac{e^{ixh}-1}{h}\,dx &= \int_{\mathbb{R}} f(x)e^{ix\xi}\lim_{h\to 0}\frac{e^{ixh}-1}{h}\,dx\\ &= i\int_{\mathbb{R}} f(x)e^{ix\xi}x\,dx\end{aligned}$$

and this last integral converges absolutely by comparison with the integral in the hypotheses.

We apply the Dominated Convergence Theorem:

$$\begin{aligned}\Big|f(x)e^{ix\xi}\frac{e^{ixh}-1}{h}\Big| &= |f(x)|\Big|\frac{\cos(xh)-1+i\sin(xh)}{h}\Big|\\ &= |f(x)|\Big|\frac{\cos(xh)-1+i\sin(xh)}{h}\Big|\\ &= |f(x)|\Big|\frac{x(\cos(xh)-1)+ix\sin(xh)}{xh}\Big|\\ &\le |f(x)||x|(M(x,h)+N(x,h))\end{aligned}$$

where M and N are bounded; thus the initial integrand is dominated by an integrable function for small values of h and so the limit and integral can be interchanged, as was needed.

Question 8 It suffices to show that $F=\nabla f$ for some C^1 function $f:\mathbb{R}^3\to\mathbb{R}$.

Consider the function

$$h(t) := \int_0^t g(s)\,ds$$

which is C^1 by the Fundamental Theorem of Calculus and the fact that g is itself C^1. Then

$$\begin{aligned}F(\mathbf{r}) &= g(x^2+y^2+z^2)(x,y,z)\\ &= \big(g(x^2+y^2+z^2)x, g(x^2+y^2+z^2)y, g(x^2+y^2+z^2)z\big)\\ &= \Big(\frac{1}{2}\frac{\delta}{\delta x}h(x^2+y^2+z^2), \frac{1}{2}\frac{\delta}{\delta y}h(x^2+y^2+z^2), \frac{1}{2}\frac{\delta}{\delta z}h(x^2+y^2+z^2)\Big)\end{aligned}$$