

Fall 2001 Qualifying Exam, Selected Solutions

DAY I

Question 4a Construct the function

$$g(z) := f(z + c) - f(z)$$

Then g is entire and yet zero on a set of points with limit point in the open set on which it is holomorphic (i.e., \mathbb{C}). Thus $g \equiv 0$ and the conclusion follows.

Question 4b The complex numbers a and b are linearly independent considered as vectors in \mathbb{R}^2 ; therefore every complex number z is of the form

$$z = \alpha a + \beta b$$

for unique α, β real. Hence $f(z) = f(\alpha a + \beta b)$, whence we can see that $0 \leq \alpha, \beta < 1$ without loss of generality. But f is entire on a compact set containing the polygon

$$\{z \in \mathbb{C} : \exists (a, b) \in [0, 1]^2 \mid z = \alpha a + \beta b\}$$

and thus $|f(z)| < M$ for some $M \in \mathbb{R}_+$ on such a square. By periodicity, then, f is both entire and bounded, hence constant by Liouville.

DAY II

Question 2 We call

$$f(z) := \frac{1}{(z^2 - 4z + 5)^2}$$

integrate $f(z)$ along the curve

$$\gamma := \gamma_1 \cup \gamma_2$$

where

$$\gamma_1 := [-R, R]$$

and

$$\gamma_2 := \{Re^{i\theta} \mid 0 < \theta < R\}$$

oriented positively, then note that the integral of γ_2 vanishes as $R \rightarrow \infty$, and then apply the Residue Theorem.

One can verify using the Quadratic Formula that the two poles, both of order two, of f occur at $2 \pm i$; hence f is meromorphic on an open set containing γ and the Residue Theorem applies. Only $2 + i$ will lie within the interior of γ ; call $\alpha := 2 + i$. By the Residue Theorem

$$\begin{aligned}
(2\pi i) \operatorname{Res}_f(\alpha) &= \int_{\gamma} f(z) dz \\
&= \int_{\gamma_1} f(x) dx + \int_{\gamma_2} f(z) dz
\end{aligned}$$

We bound the second integral above to obtain

$$\begin{aligned}
\left| \int_{\gamma_2} f(z) dz \right| &= \left| \int_0^{\pi} f(Re^{i\theta}) iRe^{i\theta} d\theta \right| \\
&\leq \int_0^{\pi} |f(Re^{i\theta})| R d\theta \\
&\leq \int_0^{\pi} \frac{R}{R^4 \left(1 - \frac{4}{R} - \frac{5}{R^2}\right)^2} d\theta \\
&\rightarrow 0
\end{aligned}$$

which implies that the value of the first integral (which is the integral asked for in the question) equals a multiple of the residue of f at α . Since $f(z) = (z - \alpha)^{-2}(z - \bar{\alpha})^{-2}$, the residue equals the first derivative of $(z - \bar{\alpha})^{-2}$ calculated at $z = \alpha$, which equals

$$-2(\alpha - \bar{\alpha})^{-3} = -2(\alpha - \bar{\alpha})^{-3}(2i)^3(2i)^{-3} = -2[\Im(\alpha)]^{-3}(2i)^{-3}$$

So the integral comes out to

$$(2\pi i)(-2)[\Im(\alpha)]^{-3}(2i)^{-3} = \pi(-2)(2i)^{-2} = \frac{\pi}{2}$$

Question 3 Call $x_0 := 0$, numerate the points in $[0, 1] \setminus S$ as

$$[0, 1] \setminus S := \{x_1, \dots, x_{k-1}\}$$

(with $i < j \Rightarrow x_i < x_j$) and call $x_k := 1$. Suppose for sake of contradiction that $f(1) < f(0)$; then in particular the set

$$\{n \in \mathbb{N} : f(x_n) < f(0)\}$$

is non-empty and has a least element called N . The function f is continuous on $[x_{N-1}, x_N]$ and differentiable on (x_{N-1}, x_N) by the hypotheses, so the MVT implies that there exists $c \in (x_{N-1}, x_N) \subset S$ such that

$$f'(c) = \frac{f(x_N) - f(x_{N-1})}{x_N - x_{N-1}}$$

The denominator is strictly positive, and the numerator must be strictly negative or else $f(0) > f(x_N) \geq f(x_{N-1})$ which is a contradiction of the leastness of N . But this, itself, gives a contradiction of the hypothesis that $f'(x)$ is positive on S .

Question 7 The fact that f is non-constant implies, by Day I Question 4b, that f cannot be entire. So f has at least one pole and, by periodicity, at least one pole in the square

$$\{z \in \mathbb{C} : -1/2 \leq \Re(z), \Im(z) < 1/2\}$$

A pole in this square can be at most distance $1/\sqrt{2}$ from the origin.

Suppose that the Maclaurin series of f has radius of convergence $r > 1/\sqrt{2}$; then the Maclaurin series defines a holomorphic function on $B(0, r)$, where f is holomorphic except at (a necessarily finite number of) points. In particular, f can be holomorphically extended to a set containing its poles, which contradicts the definition of “pole.”

Question 9 If $|r| \leq 1$ then the proposition is vacuously true; suppose $|r| > 1$. Then

$$\begin{aligned} |f(r)| = 0 &= |r^n + a_{n-1}r^{n-1} + \dots + a_0| \\ &\geq |r^n + \dots + a_1r| - |a_0| \\ &\Leftrightarrow |a_0| \geq |r^{n-1} + \dots + a_1||r| \\ &\Leftrightarrow |a_0|/|r| \geq |r^{n-1} + \dots + a_1| \end{aligned}$$

But since $|r| > 1$ we have

$$\Leftrightarrow |a_0| > |r^{n-1} + \dots + a_1|$$

We apply the same step $n - 1$ more times, without dividing by $|r|$ on the final application, to obtain

$$|a_0| + \dots + |a_{n-1}| > |r|$$

and the claim is proven.