True/False Question Answers for Chapter 6 of Spence

et. al

(erratum: should be "… of Elementary Linear Algebra by Friedberg et al.")

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6.1: The Geometry of Vectors

61. TRUE: clear from the definition on p. 363.

62. FALSE (... ~~is a vector in $\mathbb{R}^n$~~ is a *scalar* in $\mathbb{R}$)

63. FALSE. This is the definition for the *square* of the norm; to find the norm, take the dot product of the vector with itself, then take the square root.

64. FALSE: not true when the multiple is negative (see Theorem 6.1(g)).

65. FALSE: by the triangle inequality (Theorem 6.4), the norm of a sum of vectors is *less than or equal to* the sum of their norms.

66. TRUE: this is the Pythagorean Theorem (Theorem 6.2).

67. TRUE: see the diagram on p. 366.

68. TRUE. Indeed, suppose that this is false and choose $\vec{u}$ and $\vec{v}$ such that $||\vec{u}|| > 0$ and $||\vec{v}|| < 0$. Then, by the Cauchy-Schwarz Inequality (Theorem 6.3),

$$0 \leq |\vec{u} \cdot \vec{v}| \leq ||\vec{u}|| \cdot ||\vec{v}|| < 0 \tag{1}$$

implying that $0 < 0$, which is clearly false. (Note: for those interested, the statement is in fact true based solely on the definition of *norm* – see the Wikipedia page for *norm*, or come to office hours, for more information.)

69. TRUE. By Theorem 6.1(a), $||u|| = 0$ if and only if $\vec{u} \cdot \vec{u} = 0$, which by Theorem 6.1(b) is true if and only if $\vec{u} = 0$. (Aside: this is another statement that is true by the definition of *norm*).

70. FALSE: the vectors $\vec{u}$ and $\vec{v}$ may be non-zero and orthogonal.

71. FALSE: see the Cauchy-Schwartz Inequality (Theorem 6.3).

72. TRUE, by Theorem 6.1(c).

73. TRUE, by definition of *distance* (p. 361).

74. TRUE, by Theorem 6.1(f).

75. TRUE, by Theorem 6.1(d).

76. FALSE, but true if A is replaced by A^T (see p. 364).

77. TRUE. By Theorem 6.1(g), $\|-\vec{v}\| = |-1|\|\vec{v}\| = \|\vec{v}\|$.

78. FALSE, but true if the norms are replaced by the squares of norms (Theorem 6.2). For a counterexample, consider $\vec{u} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\vec{v} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$: while $\|\vec{u}\| = \|\vec{v}\| = 1$, $\|\vec{u}+\vec{v}\| = \sqrt{2}$.

79. TRUE: see the diagram on p. 366.

80. TRUE, by definition (p. 366).

6.2: Orthogonal Vectors

41. FALSE, but true for any orthogonal subset of $\mathbb{R}^n$ containing *nonzero* vectors (Theorem 6.5).

42. TRUE: see the theorem on p. 377.

43. TRUE: "Note that any set consisting of just one vector is an orthogonal set." (p. 375)

44. TRUE. By Theorem 6.5 S is linearly independent, and therefore generates a dimension n subspace of $\mathbb{R}^n$. By Theorem 4.9, S therefore generates $\mathbb{R}^n$. Thus S is a basis for $\mathbb{R}^n$.

45. TRUE: see “Representation of a Vector ... Basis” on p. 376.

46. TRUE. Apply Theorem 6.1(g) to see that the norm of this vector is 1.

47. TRUE: they form a basis, and $\vec{e}_i \cdot \vec{e}_j = 0$ for any $i < j$ (in each component, either both vectors are 0, or one is 0 and the other is 1).

48. TRUE. An orthonormal set is orthogonal and contains no zero vectors; thus Theorem 6.5 implies that an orthonormal set is linearly independent.

49. FALSE. For example, the set containing only $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is orthonormal, and the set containing only $v_2 = \begin{pmatrix} \sqrt{2}/2 \\ \sqrt{2}/2 \end{pmatrix}$ is orthonormal, yet the set containing them both is not because $v_1 \cdot v_2 = \sqrt{2}/2 \neq 0$.

50. FALSE. Take $\vec{z} = \vec{x} \neq 0$, and recall that no non-zero vector is orthogonal to itself.

51. TRUE. “[The Gram-Schmidt process] gives a method for converting any linearly independent set into an orthogonal set.” (p. 377)

52. FALSE: only R is necessarily upper-triangular.

6.3: Orthogonal Projections

33. FALSE. Although $\left(S^{\perp}\right)^{\perp}$ is a subspace, S need not be one. (But the statement *is* true if S is a subspace: see Lemma 1.)

34. FALSE. For example, let $\vec{v}$ be a non-zero vector and c a non-zero scalar. Then $\{\vec{v}\}^{\perp} = \{c\vec{\cdot}v\}^{\perp}$.

35. TRUE. This is, verbatim, the first theorem on p. 390.

36. FALSE, but true if *Col A* is replaced by *Row A*.

37. TRUE.

Lemma 1 *If S is a subspace of $\mathbb{R}^n$, then $S = \left(S^{\perp}\right)^{\perp}$.*

PROOF First of all, $S \subseteq \left(S^{\perp}\right)^{\perp}$. For if $u \in S$ then, by definition of $S^{\perp}$, $u \cdot v = 0$ for every $v \in S^{\perp}$.

Moreover, by the theorem on p. 393, $\dim S + \dim S^{\perp} = \dim \left(S^{\perp}\right)^{\perp} + \dim S^{\perp} = n$, so that $\dim S = \dim \left(S^{\perp}\right)^{\perp}$. Since $S \subseteq \left(S^{\perp}\right)^{\perp}$ and both subspaces have the same dimension, Theorem 4.9 implies that $S = \left(S^{\perp}\right)^{\perp}$, as was to be proved. ■

By Lemma 1, $(Null\,A)^{\perp} = (Row\,A)^{\perp\perp} = Row\,A$.

38. TRUE. Since every vector in $\mathbb{R}^n$ can be uniquely expressed as the sum of a vector in W and a vector in $W^{\perp}$, every vector in $\mathbb{R}^n$ can be expressed in one and only one way as a linear combination of the given vectors; thus the set is a base. It is also orthonormal, because $\vec{w}_i \cdot \vec{z}_j = 0$ by definition of $W^{\perp}$ and each of the two bases is orthonormal by itself.

39. TRUE. By Theorem 6.1(b), $\vec{0}$ is the only vector that is orthogonal to itself.

40. TRUE. By the Closest Vector Property (p. 397), $U_W(\vec{u})$ is the sole vector with this property.

41. TRUE, by definition (p. 393).

42. FALSE, by the theorem on p. 393.

43. FALSE, but true if $\{\vec{w}_1, ..., \vec{w}_k\}$ is an orthonormal basis.

44. TRUE, by definition of *distance* (p. 397).

45. TRUE, by definition of P_W (p. 395) and the Closest Vector Property (p. 393).

46. FALSE: this is not true if $W = 0$ (in which case P_W is the zero matrix of appropriate dimensions).

47. TRUE, by definition of *orthogonal projection* (p. 393) and P_W (p. 395).

48. FALSE: the basis need not be orthonormal.

49. FALSE: the columns must form a *basis* (that is, be linearly independent as well). See Theorem 6.8.

50. TRUE. If the columns form a basis, then they are linearly independent. The conclusion follows from the lemma on p. 395.

51. FALSE, for then $P_W\vec{v} = \vec{v}$, which is the projection of $\vec{v}$ onto W if and only if $\vec{v} \in W$.

52. TRUE. For P_W is uniquely defined, with its formula given by Theorem 6.8.

53. TRUE. This statement is equivalent to the statement of Question 47, which is true.

54. FALSE: it is $\|\vec{u} - P_W\vec{u}\|$.

55. TRUE. For, by the definition of P_W, the set *Null* P_W is the set of all vectors with orthogonal projection $\vec{0}$ in W. If $W = 0$, then this is the set of all vectors, and indeed $W^\perp = \mathbb{R}^n$. So suppose $W \neq \{0\}$. Then by Theorem 6.7, if $\{\vec{v}_1, ..., \vec{v}_k\}$ is an orthonormal basis for W, then the orthogonal projection of $\vec{u}$ in W is $\vec{0}$ if and only if $\vec{u} \cdot \vec{v}_i = 0$ for all $i = 1, ..., k$. But this condition implies that $\vec{u}$ is in the orthogonal complement of the basis of W, hence implying that $\vec{u} \in W^\perp$ (second theorem on p. 390). Thus *Null* $P_W \subseteq W^\perp$. And if $\vec{u} \in W^\perp$,

then the definition of *orthogonal projection* (p. 393) implies that $\vec{u} = P_W\vec{u} + \vec{u}$, hence that $P_W\vec{u} = 0$. This proves that $Null\ P_W \supseteq W^{\perp}$, and thus that $Null\ P_W = W^{\perp}$.

56. TRUE. The vector $\vec{u}$ may be uniquely expressed as the sum of a vector in W and a vector in $W^{\perp}$. But if $\vec{u} \in W$,

6.4: Least-Squares Approximations ...

28. FALSE: it minimizes the sum of the *squares* of the distances.

29. TRUE: see the equation on p. 404 directly above Example 1.

30. FALSE. The discussion in this section extends the method to polynomials of any degree.

31. FALSE. See the discussion on pp. 407-8; there are several such vectors in general, albeit a unique one of minimum norm.

32. TRUE: see "Solutions of Least Norm," p. 408.

6.5: Orthogonal Matrices and Operators

17. TRUE, by definition (p. 412).

18. FALSE. Any such operator is a rigid motion, yet only linear rigid motions are guaranteed to be orthogonal.

19. FALSE. By the first boxed (unnumbered) theorem on p. 414, a linear operator T preserves dot products if and only if T is orthogonal.

20. TRUE. By the same theorem as for Question 19, a linear operator T preserves dot products if and only if it preserves norms.

21. TRUE: this is Theorem 6.10(d).

22. TRUE: combine Theorems 6.10(b) and (d).

23. FALSE. For example, let $P = Q$; then the columns of $P + Q = 2 \cdot P$ have norm 2.

24. FALSE. For example, consider the matrix

$$P = \begin{pmatrix} -101 & 50 \\ 2 & -1 \end{pmatrix} \tag{2}$$

Then $\det(P) = (-101)(-1) - 50 \cdot 2 = 1$, yet P is clearly not orthogonal.

25. TRUE: this is Theorem 6.10(b).

26. FALSE. The basis formed must be *orthonormal*, and not merely *orthogonal*.

27. FALSE: see Example 4 on p. 396.

28. TRUE, by Theorem 6.9(a)(c).

29. TRUE, by Theorem 6.9(a)(c).

30. TRUE. If T_θ is this operator then T is linear and $\|T_\theta \vec{u}\| = \|\vec{u}\|$. Thus, by the first boxed theorem of p. 414, T is orthogonal.

31. FALSE. By Theorem 6.11(b), such operators are *reflections*, and not *rotations*.

32. FALSE, but true if the rigid motion is linear.

33. FALSE: translations are an example of non-linear rigid motions (p. 419).

34. TRUE: see p. 419.

35. TRUE: stated verbatim at the bottom of p. 419.

36. TRUE: stated verbatim at the top of p. 420.

6.6: Symmetric Matrices

21. TRUE. By Theorem 6.15 (p. 426), an $n \times n$ symmetric matrix A has a set of eigenvectors that form an orthonormal basis for $\mathbb{R}^n$. Hence Theorem 5.2 (p. 315) implies that A is diagonalizable.

22. FALSE. For any scalar c and eigenvector $\vec{v}$, the vector $c \cdot \vec{v}$ is also an eigenvector. So the columns of P need not have unit norm.

23. TRUE, by Theorem 6.15.

24. FALSE, but true for symmetric matrices.

25. FALSE. See the answer to Question 22: two distinct eigenvectors may be scalar multiples of one another, thus (because eigenvectors are non-zero) not orthogonal.

26. TRUE. "[If $b \neq 0$ in Equation 7 then] it is always possible to rotate the x- and y-axes to new x'- and y'-axes so that the major axis of the conic section is parallel to one of these new axes [implying that $b = 0$]." (p. 428)

27. TRUE. If $d \neq 0$ or $e \neq 0$, then let $\vec{v} = \begin{pmatrix} x - x_0 \\ y - y_0 \end{pmatrix}$, where (x_0, y_0) is the centre of the conic section.

28. FALSE, but see the answers to Questions 29 through 32.

29. TRUE, by the Spectral Decomposition Theorem (Theorem 6.16(a), p. 432).

30. TRUE, by the Spectral Decomposition Theorem (Theorem 6.16(a)(b), p. 432).

31. TRUE, by the Spectral Decomposition Theorem (Theorem 6.16(a), p. 432).

32. FALSE. Rotation by $\theta \pm 180^\circ$ has the same effect.

33. TRUE, by Theorem 6.14 (p. 425).

34. FALSE, but true if the eigenvalues are distinct. If they are not, then each vector $\vec{u}_i$ corresponding to a certain eigenvalue can be replaced by linear combinations of the $\vec{u}_i$, resulting in non-unique P_i (using the terms on p. 432).

35. FALSE. However, all *symmetric* matrices do have a spectral decomposition.

36. FALSE. The correct matrix is

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix} \tag{3}$$

By the derivation on p. 430, the value of d is irrelevant.

37. TRUE, by the analysis on p. 430.

38. TRUE, a consequence of Theorem 6.15.

39. FALSE. Counterexample:

$$P = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4}$$

40. FALSE: the columns of P must be chosen so that P is a rotation (and not a reflection).