Linear Optimization Review Session Summary

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February 27, 2019

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1 Canonical form

For purposes of these chapters, a problem in **canonical form** has all of the following four elements:

1. seeks the MAXIMUM value of $c^T x$
2. taken over all FEASIBLE values of x,
3. where x satisfies an EQUALITY $Ax = b$
4. and b is fixed with ALL NON-NEGATIVE COMPONENTS.

The simplex method applies only to problems in canonical form; however, problems that lack one of the above elements may usually be converted into canonical form through simple techniques.

2 Troubleshooting

NOTE: Consult the following list in sequential order.

2.1 Oh no, my problem asked me to minimize! Time to panic!

Forbear to panic, friends! For, as the old farm saying goes –

"Minimizing $c^T x$ on one hand, maximizing $-c^T x$ on the other"

So just substitute vector $-c$ for c and solve the corresponding maximization problem.

2.2 I have two or more inequalities, some pair of which face opposite directions!

Multiply both sides of the inequality by -1.[1]

2.3 Help, my program violates rule (3) of canonical form because it has [strictly more than zero] inequalities!

Add one *slack variable* per inequality. Each slack variable column should have a 1 in the matching row and 0 in all other rows.[2]

Replace each inequality sign with an equality sign.

2.4 The column b has negative entries!

After adding slack variables as described above, multiply all rows i with $b_i < 0$ by -1.

****At this point you should have an optimization problem satisfying prongs 1-3.****

To ensure an initial BFS, however, you must add *artificial variables* at this point. In general, you need one per row of A; however, you

[1] NOTE: It is not mathematically incorrect to omit this step *as long as* you remember to *subtract* the corresponding slack variables.

[2] assuming that all your inequalities are $\leq$; otherwise, replace 1 with -1 in any columns with $\geq$.

may omit the a.v. for a row if that row already contains an entry for a suitable basic variable (see the example, next section,).

3 Problem 6 of the review

3.1 Walkthrough

1. [OPTIONAL] Multiply the first ineq. by -1 and switch direction.
2. Add slack variables x_5 and x_6 for, respectively, the inequalities with positive and negative rhs.
3. If you performed step 1 then the columns for x_5 and x_6 should both have a single 1 entry; if you skipped it, make sure that the column for x_5 contains a -1 and not a 1.
4. Since we have prongs (1)-(3) but not (4) of canonical form, add artificial variables. We need only ONE; because columns 3 and 5 are identity matrix columns, we may set
   - $x_3 = 10$ and
   - $x_6 = 12$

   and already have two of the three non-negative components needed for our initial basic feasible solution. Hence we need just one more candidate, y_1, which we may set to 10.
5. Draw a tableau – *excluding the objective row* – that includes all variables. Label each column with its corresponding variable and each row with its corresponding basic variable – all of which will initially be slack or artificial.
   ****At this point, all but the objective row of your tableau should be identical to the one you see on the exam review key. (Note that the tableau I drew during the review session was off by the sign of the y_1 entry – I will get to that shortly.)****
6. Remember that Phase I consists of minimizing (*i.e.,* maximizing the *negative* of) the sum of the artificial variables. Since we have only one of those, y_1, the objective function to maximize at this phase is

$$z = -y_1 = 2x_1 - 2x_2 - x_4 - 10$$

an equation we obtained from the third row of the tableau. Subtracting the x_i from the rhs of this equation gives us the objective row in the key (note in particular that there is no y_1 term!).

3.2 Potential pitfalls, errata

All expressions should be interpreted in the scope of review question #6 and the previous subsection unless otherwise specified.

- It is indeed legal to turn inequalities with $\geq$ into equations by *subtracting* a slack variable. I'm just used to dividing by -1 and then adding. HOWEVER, you must not forget to subtract rather than add.
- When in doubt (for our current purposes) don't omit an "extraneous" variable. Done right, omitting might eliminate redundancy and save a little time; done wrong or haphazardly it will lead to a wrong answer.
- **(follow-up from the session)** Column **b** should be made non-negative BEFORE any artificial variables are added. Doing the reverse (i.e., adding a.v.s and THEN making $\mathbf{b} \geq 0$) defeats the point of artificial variables, rendering them no different than slack variables. (Make sure you understand the difference between the *purposes* of slack and artificial variables.)

  Note that I reversed the order of these steps during the review session, resulting in a switched sign on the third row in the y_1 column.
- The variable x_5 canNOT be substituted for x_3 in the initial BFS – doing so would require setting $x_5 = 10 < 0$. (This fact is tied to the reason why we need artificial variables – loosely speaking, we run out of candidate basic variables to set to non-negative values.)
- Near the end of the session, I said that the last row of the answer key tableau was incorrect. It isn't. I don't know why I said that, but I was probably really tired or something. Sorry.

{ best of luck! }