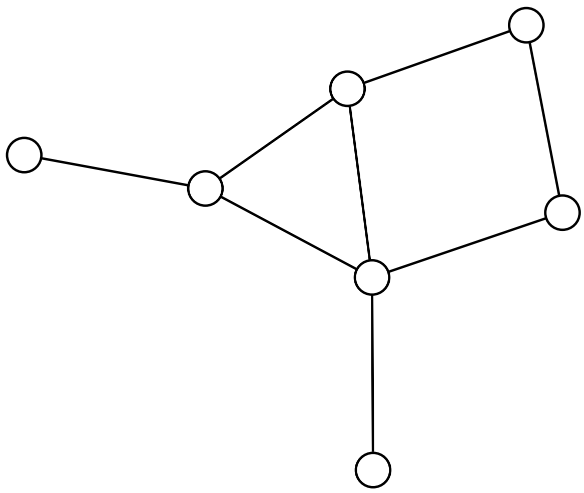
Quiz 2 Solutions

Question 1 Consider the following graph G:



1. Is G bipartite? If so, provide a partition of its vertices into appropriate sets; if not, rigorously explain why not.

   Answer No. A graph is bipartite if and only if it contains no odd cycles, and G contains an odd cycle.

2. Define the *diameter* of a graph and find the diameter of G.

   Answer The *diameter* of G is the maximum value of $d(x, y)$ for x, y in G; for G this value is 3.

3. Find the quantities $\delta(G)$ and $\Delta(G)$.

Answer These values indicate the minimum and maximum degree, respectively, so $\delta(G) = 1$ and $\Delta(G) = 4$.

4. Give the order and size of G.

Answer The order is 7 and the size is 8.

Directions Answer AT LEAST ONE of questions 2 and 3. If you attempt to answer both, clearly indicate which one should be graded first: if you earn at least 80% credit on this one then points for the other will count for extra credit in the Quizzes category.

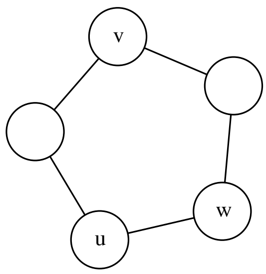
***

Question 2 Prove or disprove that each of the following relations is an equivalence relation:

- On the set $\mathbb{Z}$ of integers, $m \sim n$ if and only if m and n are either both odd or both even (*i.e.,* iff they have *the same parity*)
- On the set $V(G)$ of vertices in an arbitrary graph G, $u \sim v$ if and only if u and v are NOT adjacent.

Answer

- For any integer m, $m \sim m$ trivially. We have $m \sim n \to n \sim m$; it doesn't matter what order the integers are given in. Finally, to prove transitivity, consider that two integers m and n have the same parity if and only if $m - n$ is even. So if $m \sim n$ and $n \sim p$ then $m - n$ is even and $n - p$ is even, so $m - p$ is even and thus $m \sim p$. Because it satisfies all three prongs, $\sim$ is an equivalence relation.
- No vertex can be adjacent to itself in a graph, and if $u \not\sim v$ then $v \not\sim u$. But the relation is not transitive: for example, consider the following graph:



We have $u \not\sim v$ and $v \not\sim w$ in the graph, yet $u \sim w$.

Question 3 Recall that the graph $G[S]$ is the induced subgraph of G on the subset S of $V(G)$. Prove that the vertices of every connected graph of order n can be labelled in an order

$$\{v_1, ..., v_n\}$$

such that the induced graph

$$G[v_1, ..., v_k]$$

is connected for every k with $1 \leq k \leq n$.

Answer We simultaneously prove two things by induction: (1) that we can "greedily" choose vertices at each step and (2) that this choice results in a connected induced graph for all k.

(Base case.) Let v_1 be arbitrary in G. Then $G[v_1]$ is trivial and thus connected.

(Induction step.) For $k \geq 1$ such that $k < n$, assume the theorem to be true. Then there exists some $v_{k+1} \in G \backslash \{v_1, ..., v_k\}$ that is adjacent to a vertex in $G[v_1, ..., v_k]$. Once we prove this claim, we will have proven that we can find a v_{k+1} such that $G[v_1, ..., v_k]$ is connected and our proof will be complete.

The proof of the claim uses the connectedness of G. Indeed, take a point x in the induced graph and a point y outside it and draw an $x - y$ path: because y is not in the induced graph, this path contains an edge between a vertex in the induced graph and a vertex outside.