Quiz 1 Solutions

Question 1 Suppose G is a connected graph of order n. What are the greatest and least possible sizes of G? Fully justify both answers.

Solution Recall that the **size** of G is the size of $|E(G)|$. Since $|E(G)|$ is a subset of the 2-subsets of $V(G)$, of which there are $\binom{n}{2}$, the largest possible size of G is $\binom{n}{2}$.

Finding the least possible size is trickier. The graph P_n has size $n-1$, for example, but it may not be immediately clear how to show that the bound is *sharp* – *i.e.,* that **any** graph of size $n-2$ must be disconnected.

Multiple approaches are possible: one example that does not use induction is the following. Suppose G is connected. Fix some $x \in G$ and write the $x-u$ geodesic, for $x \neq u$, as

$$xx_1...x_{k-1}x_k$$

where $x_k = u$ and k is the distance between x and u, possibly 1.

By connectivity, each of the $n-1$ vertices u of G other than x has an $x-u$ geodesic, each of which has a terminal edge $x_{k-1}x_k$.

Suppose two geodesics have the same terminal edge $\{uv\}$: *i.e.,* one geodesic terminates

$$...uv$$

and the other terminates

$$...vu$$

Then neither u nor v equals x (otherwise the geodesic would have length 0). If $d(x,v) = k$, then $d(x,u) = k-1$ from the position of u in the first geodesic. But then the position of u in the second geodesic implies $d(x,v) = k-2$, contradiction. Thus no two $x-u$ geodesics have the same terminal edge, and since there are $n-1$ $x-u$ geodesics, any connected graph must have at least $n-1$ edges.

We combine the two facts proved above – namely, that

1. there exists a connected graph of order n and size $n-1$, and
2. no connected graph of order n can have size *strictly less than* $n-1$

to conclude that the minimum size for a graph of order n is $n-1$. ■

Question 2 Prove that every graph G has a path of length at least $\delta(G)$.

HINT: Let $P = x_1x_2...x_k$ be the **longest path** in G. Combine an inequality between $\delta(G)$ and $|\Gamma(x_k)|$ with the fact that P *cannot be extended.*

Solution **Every element of $\Gamma(x_k)$ is on the path P.** To see why, suppose that this were not so and let y be a vertex adjacent to x_k but not on P. Then P could be extended to $x_{k+1} = y$, forming a path of length $k+1$ and contradicting the assumption that P was the longest path.

This last statement shows that $\Gamma(x_k)$ is a subset of P, which implies that $|\Gamma(x_k)| \leq k$. Since $\delta(G) \leq |\Gamma(x_k)|$, by transitivity we obtain the inequality

$$\delta(G) \leq k$$

Since k is the length of the *longest* path in G, it follows that at least one path has length at least $\delta(G)$. ■