Question 1 Classify the following statements as TRUE (T) or FALSE (F). No justification is necessary.

Note that *true* means *true without further conditions.*

1. ________ Every walk is a path.
   False: a path is a walk with distinct vertices.

2. ________ Every path is a trail.
   True: a path is a trail with distinct vertices.

3. ________ A graph is a forest if and only if each of its edges is a bridge.
   True. A bridge is an edge e of a *component* K such that $K - e$ is disconnected. An edge is a bridge if and only if it lies on no cycle. So all edges of a graph are bridges if and only if the graph contains no cycles – *i.e.,* is a forest.

4. ________ If a statement is false, then its contrapositive is false.
   True.

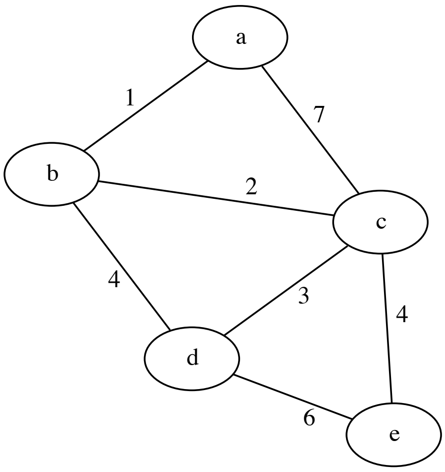
5. ________ A graph is bipartite if and only if it contains at least one odd cycle.
   False. The opposite is true: a graph is bipartite iff it *lacks* odd cycles.

6. ________ Every tree is bipartite.
   True because trees are acyclic and therefore contain no odd cycles.

7. ________ Every bipartite graph is connected.
   False. Counterexample: $P_2 \cup P_2$.

Question 2

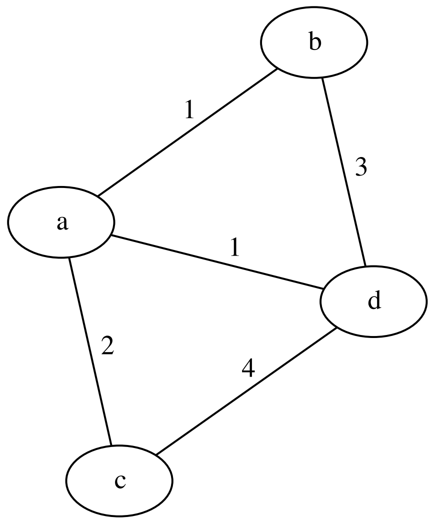
(a) Use Kruskal's algorithm to find a minimal spanning tree of the following graph, listing the edges of the spanning tree in the order you add them to the tree:



Answer

1. $\{a, b\}$
2. $\{b, c\}$
3. $\{c, d\}$
4. $\{c, e\}$

(b) Use Prim's algorithm, starting at vertex b, to find a minimal spanning tree of the following graph, listing the edges of the spanning tree in the order you add them to the tree:



Answer

1. $\{a, b\}$
2. $\{a, d\}$
3. $\{a, c\}$

(c) Repeat part (b), except starting at vertex d.

Answer

1. $\{a, d\}$
2. $\{a, b\}$
3. $\{a, c\}$

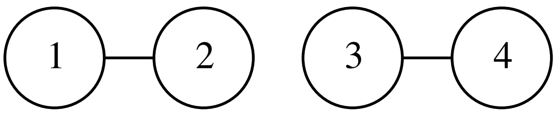
Question 3 Consider the following theorem and its proof:

> **Theorem** Let G be a graph with more than one vertex. If $\delta(G) = 0$ then G is disconnected.
>
> PROOF Let u be the vertex of degree zero and v any other vertex. If there were a $u - v$ path then there would be an initial $u - w$ edge for some $w \neq u$ in the path, but u is incident to no edges. Contradiction.

State the *converse* of the above theorem, and then prove that the converse is FALSE.

Converse If G is disconnected for G such that $|G| > 1$ then $\delta(G) = 0$.

PROOF The following graph is disconnected and of order four, yet contains no vertices of degree zero:



$\blacksquare$

Question 4

(a) For a tree T, let $\rho(T)$ be defined as follows:

$$\rho(T) := \frac{|\{v \in T : d(v) = 1\}|}{|T|}$$

- Find (and justify) maximum and minimum values for $\rho(T)$ over the set of all trees. (We assume, as always, that no graph has order zero.)
- Find two sequences, $\{S_n\}$ and $\{T_n\}$, of trees with the property that, as $n \to \infty$, $|S_n| \to \infty$ and $|T_n| \to \infty$ yet $\rho(S_n) \to 0$ and $\rho(T_n) \to 1$.

Answer In plain English, the numerator is the number of leaves of T and the denominator is the order of T. In other words, ρ gives the proportion of the vertices in T that are leaves, thus certainly

$$0 \leq \rho(T) \leq 1$$

Note that this inequality is not good enough standing alone. We have to find actual minimum and maximum values and justify them; it turns out, however that those values are 0 and 1 respectively.

Indeed, the minimum is attained with a tree of order 1 (the trivial graph), the only tree with fewer than two leaves: this tree T_1 has a single vertex of degree zero – hence zero leaves – and thus $\rho(T_1) = 0$. On the other hand, drawing from the fact that all trees of order at least two have two or more leaves, we take T_2 to be the tree of order 2 and observe that $\rho(T_2) = 1$.

Define S_n and T_n as follows:

$$\begin{aligned}
V(S_n) &:= V(T_n) := \{v_1, ..., v_n\} \\
E(S_n) &:= \{\{v_i, v_{i+1}\} : i \in \{1, ..., n-1\}\} \\
E(T_n) &:= \{\{v_1, v_i\} : i \in \{2, ..., n\}\}
\end{aligned}$$

■

In the below, $a \mid b$ signifies "*a divides b*."

(b) A modified version of *Euclid's lemma*, which you are not required to prove, reads as follows:

> Let p and $p_1, ..., p_k$ be prime numbers; also let $p \mid (p_1 \cdot ... \cdot p_k)$. Then $p = p_i$ for some $i \in \{1, ..., k\}$.

Complete the following proof by contradiction that infinitely many prime numbers exist:

PROOF Suppose that there were finitely many (say, a total of k) prime numbers. Then we could enumerate them as $2 = p_1 < p_2 < ... < p_k$. Take the product of these k prime numbers and call it N.

HINT: You may assume that if $p \mid n$ then $p \nmid (n + 1)$
(*i.e.,* p does *not* divide ...).

Answer Then, by the modified version of the Fundamental Theorem of Arithmetic we learned in class, $N + 1$ can be expressed as a product of solely prime numbers (*i.e.,* , can be *decomposed* into a product of primes), thus $p_i \mid (N + 1)$ for some $i \in \{1, ..., k\}$. On the other hand, this same p_i divides N by Euclid's lemma. This fact poses a contradiction because a prime cannot divide two consecutive natural numbers. ■

(c) Categorize each of the following degree sequences as *graphical* or *non-graphical*. Draw a graph corresponding to each graphical sequence; for each non-graphical sequence, explain how you know that the sequence is not graphical.

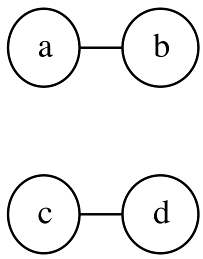
- 2, 2, 3, 3, 4, 4 **Graphical**. Using a theorem from the course and setting this sequence equal to s, the sequence s' is

  $$3, 2, 2, 1, 2$$

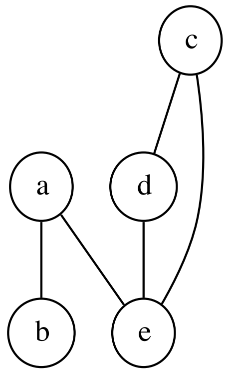
  and the sequence s'' is

  $$1, 1, 1, 1$$

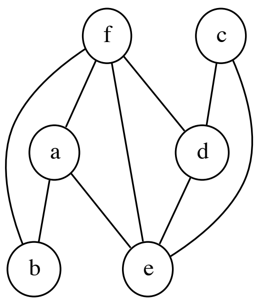
  The sequence s'' yields

  whence we find a graph for s':



and finally s:



- 4, 3, 4, 3, 2, 1
  Non-graphical by the Handshake Lemma (odd number of odd vertices).

- 1, 2, 3, 2, 5, 5
  Non-graphical. For example, this graph of order 6 has two vertices of order 5, so any vertex not equal to these two must have degree at least two. Yet there is a vertex of degree 1.

Question 5

(a) For G and H as pictured below, draw the following graphs:

- $G \times H$
- $G + H$

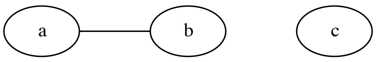


Figure 1: Graph G

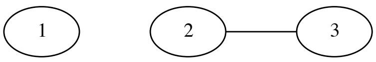


Figure 2: Graph H

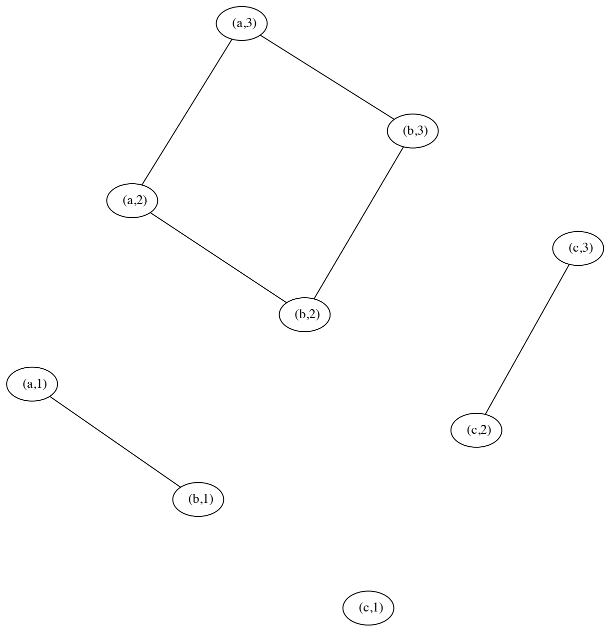


Figure 3: $G \times H$

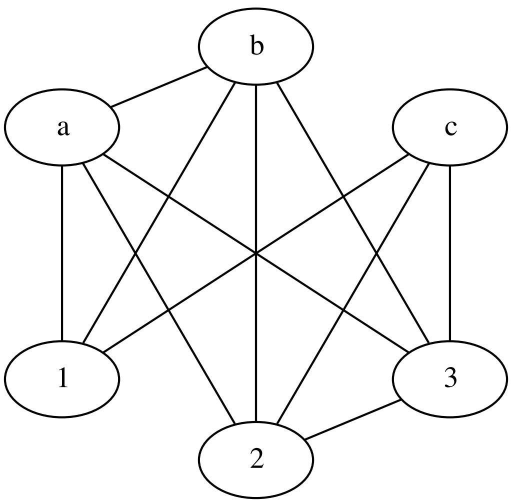


Figure 4: $G + H$

(b) Prove or disprove that the following relations R are equivalence relations:

- In a connected graph G, uRv if and only if u and v are adjacent.
- In an arbitrary graph G, uRv if and only if u and v are connected by a path.

Answer

- No; reflexivity fails because no vertex is adjacent to itself.
- Yes. Indeed, the relation is
  - **reflexive** because every vertex is connected to itself by the empty path;
  - trivially **symmetrical**; and
  - **transitive** by virtue of the fact that if uRv and vRw then there certainly exists a $u - w$ *walk*, and therefore a $u - w$ path by a theorem of Chapter 1.

(c) Find the *sizes* of the following graphs:

- A forest with a single component and no vertices of degree exactly one
  Answer: $\boxed{0}$. The forests with a single component are precisely the trees, and every tree of order greater than one has two or more leaves. Thus this forest is simply the trivial graph.

- The complete graph K_n, for $n \geq 3$
  Answer: $\boxed{\binom{n}{2}}$, because every possible 2-subset of $[n]$ is an edge.

- The *complement* of C_n (*i.e.*, , the cycle on n vertices labelled $1, ..., n$).
  Answer: $\boxed{\binom{n}{2} - n}$.

- A connected graph G of order 100, all of whose edges are bridges
  Answer: $\boxed{99}$, because a connected graph all of whose edges are bridges is acyclic and therefore a tree.

- A graph G with degree sequence $0, 2, 2, 2, 1, 2, 1$
  Answer: $\boxed{5}$, by the Handshake Lemma.