Name (please PRINT):

Out of fairness to your classmates, please stay on this page and <u>DO NOT BEGIN</u> until told otherwise.

Sloppy handwriting increases the chance of grading errors: please write from TOP TO BOTTOM, moving columns of work from LEFT TO RIGHT with STRAIGHT MARGINS in between. Ensure that no work is overlooked by clearly marking any point at which you make an exception to these guidelines.

- Close bags and silence electronics – this quiz is closed-resource.
- If you are still working when time is called, you must stop immediately and bring your quiz to the front. **<u>Absolutely no writing</u>** after time is called.
- Write your printed name on all sheets containing work.
- $\boxed{\text{Box}}$ your final answers.
- As much as possible, rubrics are designed so that your grade will not "cascade" from a mistake early in a problem: move on if you have trouble for too long in an early subproblem.
- While you generally need not write in short essay form, you must demonstrate knowledge of course material, supplementing your mathematical notation with words if necessary. In particular, you must
  - *explicitly* cite any theorems you use from the course and
  - write conclusions using at least a few words.

Final Exam, Part I
80 minutes

Score: (curved, out of 55)

Prob. No.	Grade?	Earned	Total
A	(obligatory)		14
B	(obligatory)		15
C1			13
C2			13
C3			13
C subtotal			26
Part I Total			55

Question I (A)

Categorize each of the following statements as TRUE or FALSE, where TRUE means "true without further restrictions."

1. ________ A tree is a connected forest. **T**
2. ________ Every tree is perfect. **T**
3. ________ Every complete graph is perfect. **T**
4. ________ The line graph of a graph G has order $E(G)$. **T**
5. ________ The line graph of a graph G has size $|G|$. **F**
6. ________ Containing a Hamiltonian path is a *necessary* condition for a graph to be Hamiltonian. **T**
7. ________ Containing a Hamiltonian path is a *sufficient* condition for a graph to be Hamiltonian. **F**

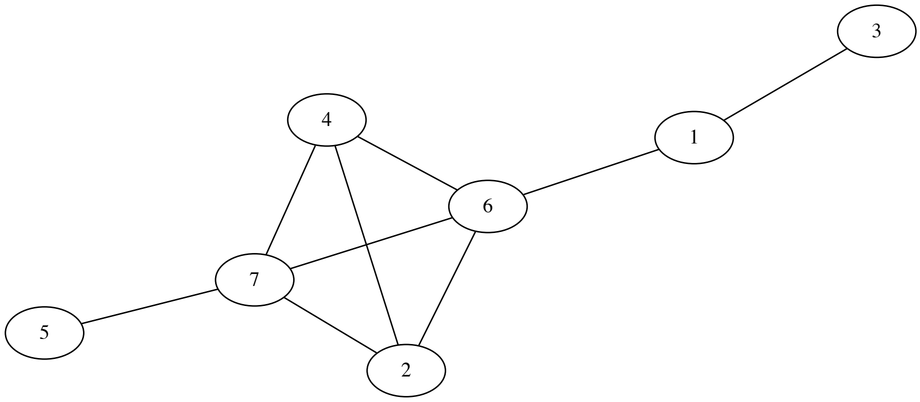
Question I (B)

During a certain five-day week (Monday through Friday), a small motel has the following reservations on its books:

Reservation No.	For nights
1	Mon - Tue (2 nights)
2	Wed - Thur (2 nights)
3	Mon (1 night)
4	Wed (1 night)
5	Fri (1 night)
6	Tue - Thur (3 nights)
7	Wed - Fri (3 nights)

Use methods of graph theory to determine the minimum number of rooms that must be reserved to accommodate these seven reservations. Assume that the same room may be booked for two consecutive nights for different parties (*i.e.,* there need not be any gap for cleaning between different stays) and that no other reservations play a relevant role.

Answer Draw a graph in which vertices represent reservations and colors represent rooms. We may assign the same room to two reservations if and only if they share no nights in common; hence, we draw an edge between exactly those vertices corresponding to reservations with at least one night in common (cf. opposite page). The graph has a 4-clique and therefore has a chromatic number of at least 4. On the other hand, its maximum degree is 4 and it is neither a complete graph nor an odd cycle, implying that its chromatic number is at most 4. Therefore, its chromatic number is exactly 4 and we require a minimum of four rooms for these reservations.



Question I (C)

Choose TWO of the following THREE questions to answer, clearly indicating your choice both below and on the score sheet. You may use the remaining blank sheets for work.

1. ________ Prove that a bipartite graph with non-empty parts L and R has a perfect matching if both of the following conditions hold:

   - $|S| = |\Gamma(S)|$ for $S \subseteq L$ such that $|S| < \frac{|L|}{2}$
   - $\Gamma(S) \neq \Gamma(T)$ for all distinct $S, T \subseteq L$

   (HINT: $\Gamma(\varnothing) = \varnothing$; also, $\varnothing \subseteq X$ for all sets X.)

2. ________ Let G be a connected graph of order at least three and the statement P_1 be defined as

   Graph G has a cut-vertex.

   and the statement P_2 be defined as

   Graph G has a bridge.

Select the strongest possible correct statement from the choices below and justify your choice.

A Neither P_1 nor P_2 implies the other.

B If P_1 then P_2.

C If P_2 then P_1.

D P_1 if and only if P_2.

Furthermore, if you select one of statements A-C, strengthen whichever direction(s) of statement D fail(s) to apply in general to yield (a) correct statement(s) [For example, if you choose C then impose some further condition that, combined with P_1, implies P_2].

3. ________ Find the quantities indicated in the table with respect to the graph G of Figure 1 [opposite page]. (Here, $d(G)$ indicates the average (arithmetic mean) of the degrees of the vertices of G.)

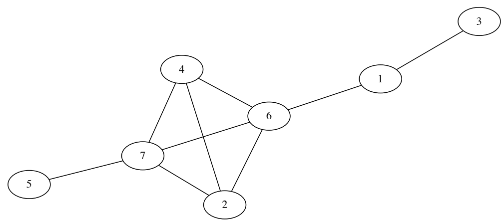


Figure 1: The graph G.

$\kappa(G)$	
$\lambda(G)$	
$\chi(G)$	
$\chi'(G)$	
$d(G)$	

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Please make sure your solutions are complete before time is called.

YOU CANNOT COME BACK TO THIS PART AFTER THE BREAK.

1 I(C) 1

Choose L to be the (not nec. strictly) larger of the two parts of the graph. First of all, the second prong implies that no vertex in L is isolated, for no vertex may share a neighbourhood with the empty set. If $|L| \geq 3$, then the first prong implies that every vertex has degree 1 and the second implies that no two vertices share the same neighbour. Thus $|S| \leq |\Gamma(S)| \leq |R|$ for every subset S of L, implying that a complete matching exists; by the assumption that $|L| \geq |R|$, this matching is perfect.

Now suppose that $|L| < 3$. If $|L| = 1 = |R|$ then a complete matching exists trivially, so suppose that $|L| = 2$. We cannot have $|R| = 1$ because of prong 2, thus $|L| = |R| = 2$. Furthermore, both vertices of L have degree exactly one and different neighbours, implying Hall's condition and therefore the existence of a perfect matching.

2 I(C) 2

Statement C is the strongest applicable statement. Indeed, $\kappa(G) \leq \lambda(G)$ in general and thus $1 \leq \kappa(G) \leq \lambda(G) = 1$ for connected graphs with a bridge; however, it is easy to find examples of graphs with cut-vertices that lack bridges.

Since $\kappa(G) \leq \lambda(G) \leq \delta(G)$, a sample correct response would be: If G has a cut-vertex *and a leaf*, then it has a bridge.

3 I(C) 3

Both $\kappa(G) = \lambda(G) = 1$ because the graph has a leaf. The chromatic number is 3; its clique number gives a lower bound of 3 and a greedy colouring an upper bound. The edge-chromatic number is 5. Finally, the average degree is simply the sum of degrees (18) over the order (7).