1 Solutions to quadratic equations

1.1 The Quadratic Formula

Theorem 1 (Quadratic Formula): The equation

$$\boxed{ax^2 + bx + c = 0}$$

with real or complex coefficients (and $a \neq 0$) has the two solutions

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Proof. We start by simply rearranging:

$$ax^2 + bx + c = 0 \tag{1}$$

$$ax^2 + bx = -c \tag{2}$$

$$x^2 + \frac{b}{a}x = -\frac{c}{a} \tag{3}$$

Remembering the form $(x+y)^2 = x^2 + 2xy + y^2$, we add $y^2 = (\frac{b}{2a})^2$ to both sides

$$x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} = -\frac{c}{a} + \frac{b^2}{4a^2} \tag{4}$$

collapse the left-hand side, and collect the terms of the right-hand side under one denominator:

$$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \tag{5}$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \tag{6}$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{7}$$

CQFD.

1.2 Nature of quadratic roots

Definition 1 (discriminant): In equation 7, the quantity $b^2 - 4ac$ is called the *discriminant* of the boxed quadratic equation. ♣

If a, b, and c are all real (which we will assume for this course), then the discriminant determines whether we have two distinct real solutions, a single real solution with multiplicity two, or two *complex conjugate* solutions.

Definition 2 (complex conjugate): Of a complex number $z = a+bi$, where a and b are real and $i^2 = -1$, the number

$$a - bi$$

Signified with $\overline{z}$. ♣

1.3 Applications to differential equation solving

Definition 3 (eigenvalue/eigenvector): Of a matrix A, a pair (λ, v_λ), with $v_\lambda \neq 0$, such that

$$Av_\lambda = \lambda v_\lambda$$

or, equivalently,

$$det(A - \lambda I) = 0$$

♣

The characteristic polynomial for any 2x2 matrix expands to

$$\lambda^2 - T\lambda + D$$

where T signifies the trace of A and D its determinant; thus the discriminant of the characteristic polynomial of any A is simply $T^2 - 4D$. Since we know whether we have distinct real, non-distinct real, or complex conjugate eigenvalues just by the *sign* of the determinant, then, this previous fact allows us to understand the nature of solutions to matrix equations without necessarily needing to calculate those solutions.

The sign of T is similarly important because it

1. provides the real part of complex eigenvalues and
2. provides the median/mean of two real eigenvalues.

So if our eigenvalues are complex and $T < 0$ we will have a spiral sink. (What if $T > 0$? $T = 0$?)

Real eigenvalues of A are non-distinct if and only if (T, D) lies on the parabola $D = T^2/4$. Otherwise, we have the following (non-degenerate) cases:

$D < 0$	one eigenvalue of each sign
$D > 0, T < 0$	two negative eigenvalues
$D > 0, T > 0$	two positive eigenvalues

Springs The behaviour of unforced springs is governed by the second-order DE

$$my'' + by' + ky = 0$$

whose characteristic polynomial has discriminant $b^2 - 4mk$. Underdamped, overdamped, and critically damped oscillators correspond to the discriminant being less than, greater than, and equal to zero, respectively.

2 Examples

2.1 Complex matrix equation

Question Solve the IVP of the matrix equation

$$Y' = AY; \quad Y(0) = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

where

$$A = \begin{pmatrix} 1 & -2 \\ 3 & -3 \end{pmatrix}$$

Answer The trace of A is -2 and the determinant of A is 3, so its characteristic polynomial is

$$x^2 + 2x + 3$$

(Verify why I was able to draw this conclusion if you aren't sure – or ask me.) So our eigenvalues are

$$x = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 3}}{2} = -1 \pm i\sqrt{2}$$

We choose the eigenvalue with *positive imaginary part* (for a reason that will be explained later) and find its corresponding eigenvector. In other words, we solve the equation

$$(A - (-1 + \sqrt{2}i)I)v = \begin{pmatrix} 2 - \sqrt{2}i & -2 \\ 3 & -2 - \sqrt{2}i \end{pmatrix} \begin{pmatrix} 1 \\ x \end{pmatrix} = 0 \tag{8}$$

Note that this equation is equivalent to the equation

$$\begin{pmatrix} 2 - \sqrt{2}i \\ 3 \end{pmatrix} + x \begin{pmatrix} -2 \\ -2 - \sqrt{2}i \end{pmatrix} = 0 \tag{9}$$

Because of the way math works, assuming that our eigenvalue is correct it is sufficient to solve for x for just one of these rows:

$$(2 - \sqrt{2}i) - 2x = 0$$
$$\frac{2 - \sqrt{2}i}{2} = x$$
$$x = 1 - \frac{\sqrt{2}}{2}i$$

So, multiplying each entry by 2, we obtain the eigenvalue and -vector pair

$$\lambda = -1 + \sqrt{2}i, \quad v = \begin{pmatrix} 2 \\ 2 - \sqrt{2}i \end{pmatrix}$$

giving us the solution form

$$e^{(-1+\sqrt{2}i)t} \begin{pmatrix} 2 \\ 2 - \sqrt{2}i \end{pmatrix} = e^{-t} \left(\cos(\sqrt{2}t) + i \sin(\sqrt{2}t)\right) \begin{pmatrix} 2 \\ 2 - \sqrt{2}i \end{pmatrix} \tag{10}$$

$$= e^{-t} \begin{pmatrix} 2\cos(\sqrt{2}t) + 2i\sin(\sqrt{2}t) \\ 2\cos\left(\sqrt{2}t\right) + \sqrt{2}\sin\left(\sqrt{2}t\right) + i\left(-\sqrt{2}\cos\left(\sqrt{2}t\right) + 2\sin\left(\sqrt{2}t\right)\right) \end{pmatrix} \tag{11}$$

yielding the general solution

$$Y(t) = e^{-t} \left[A \begin{pmatrix} 2\cos(\sqrt{2}t) \\ 2\cos(\sqrt{2}t) + \sqrt{2}\sin(\sqrt{2}t) \end{pmatrix} + B \begin{pmatrix} 2\sin(\sqrt{2}t) \\ -\sqrt{2}\cos(\sqrt{2}t) + 2\sin(\sqrt{2}t) \end{pmatrix} \right] \tag{12}$$

(Note that, if we had chosen the eigenvalue with the negative imaginary part, we would have had to negate the negative part of the expansion of Euler's Formula above, increasing the chance of sign errors.)

To find the particular solution, we plug in $t = 0$:

$$Y(0) = A \begin{pmatrix} 2 \\ 2 \end{pmatrix} + B \begin{pmatrix} 0 \\ -\sqrt{2} \end{pmatrix} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

and obtain

$$A = 0, \quad B = -\frac{3}{2}\sqrt{2}$$

$$Y(t) = -\frac{3}{2}\sqrt{2}e^{-t} \begin{pmatrix} 2\sin(\sqrt{2}t) \\ -\sqrt{2}\cos\left(\sqrt{2}t\right) + 2\sin\left(\sqrt{2}t\right) \end{pmatrix}$$

Note that this solution has amplitude diminishing with time, a fact corroborated by the fact that $T = -2 < 0$.

2.2 Second-order equation

Question Classify the oscillator governed by

$$y'' - 3y' + 2y = 0; \quad y(0) = 1, v(0) = -1$$

Also give its solution and describe its end behaviour.

Answer The characteristic polynomial has discriminant $(-3)^2 - 4 \cdot 2 \cdot 1 = 1 > 0$ and $b = 3 \neq 0$, so this oscillator is overdamped. The homogeneous solution is

$$\begin{aligned}
y(t) &= Ae^{2t} + Be^{t} \\
&= 2Ae^{2t} + Be^{t} \\
y(0) &= \quad A + B = 1 \\
-v(0) &= -2A - B = 1 = y'(0) \\
&= -A = 2 \\
&\boxed{A = -2} \\
&\boxed{B = 3}
\end{aligned}$$

Thus $\boxed{y(t) = -2e^{2t} + 3e^{t}.}$ and $\boxed{v(t) = y'(t) = -4e^{2t} + 3e^{t}.}$ As $t \to \infty$, $y(t)$ behaves like $-2e^{2t}$ and $v(t)$ like $-4e^{2t}$.

Finally, know that the corresponding matrix formula is obtained by the following process –

$$\begin{aligned}
y' &= v \\
0 &= y'' - 3y' + 2y \\
&= v' - 3v + 2y \\
3v - 2y &= v'
\end{aligned}$$

which yields this matrix:

$$\begin{pmatrix} y' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} y \\ v \end{pmatrix}$$