Review Session Notes

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1 Existence and Uniqueness Theorems

Both theorems pertain to an *initial value problem* (IVP):

$$\frac{dy}{dt} = f(t, y) \tag{1}$$

$$y(t_0) = y_0 \tag{2}$$

1.1 Existence

All mathematical expressions (such as f) in this subsection refer to those in the IVP above.

Technical definition If f is continuous in any rectangle R around (t_0, y_0) then there exist $\epsilon > 0$ and $y(t)$ such that $y(t)$ solves the IVP for $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Intuitive definition If f is continuous near (t_0, y_0) then IVP has a solution in a small time interval around t_0.

1.2 Uniqueness

Technical definition If f AND $\frac{\delta f}{\delta y}$ are continuous in any rectangle R around (t_0, y_0) AND there exists some $\epsilon > 0$ and two solutions $y_1(t)$ and $y_2(t)$ of the IVP on $t \in (t_0 - \epsilon, t_0 + \epsilon)$, then $y_1 = y_2$ on $t \in (t_0 - \epsilon, t_0 + \epsilon)$.

Intuitive definition If f is *well-behaved* near (t_0, y_0) then any two solutions to the IVP are identical near (t_0, y_0).

1.3 Important notes

- Do not worry about the size of the rectangle R as long as one exists that strictly contains the point.
- Recall from geometry that the **inverse** (equivalent to the *converse*) of a true statement need not be true; for example, while the statement

  If a polygon is a square, then the same polygon is a rectangle.

  is obviously correct, its inverse statement,

  If a polygon is NOT *a square, then the same polygon is* NOT *a rectangle.*

  is false.

  Likewise, the fact that the Existence Theorem (*resp.*, the Uniqueness Theorem) *fails* to apply at a point does *not* necessarily mean that no solutions exist coinciding with that point (*resp.*, that a solution coinciding with the point is non-unique). To see this fact for yourself, try solving the following IVP:

  $$\frac{dy}{dt} = -\frac{t}{y}; \quad y(2) = 0$$

- Consider the DE

  $$\frac{dy}{dt} = y/t^2$$

  and confirm that the function family

  $$y_c(x) = \begin{cases} 0 & t \leq 0 \\ c \cdot e^{-1/t} & t > 0 \end{cases}$$

  forms a set of solutions to the DE, where $c \in \mathbb{R}$. (If it will not sidetrack you, also confirm that these functions are continuous.)

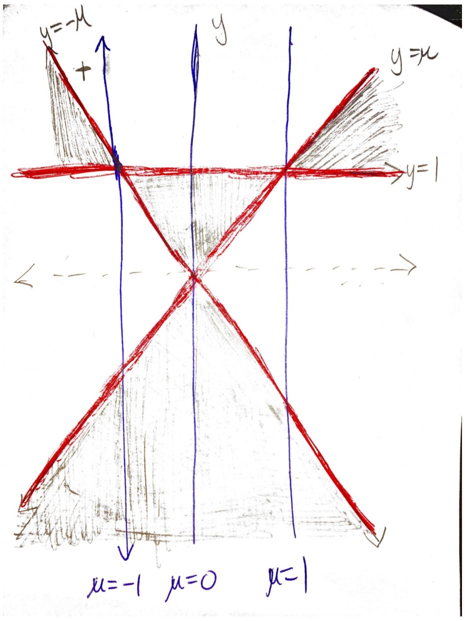
  Let $T_- = \{(t_0, 0) : t_0 < 0\}$. Since $f(t, y) = y/t^2$ and $f_y = 1/t^2$, the Uniqueness Theorem applies at every point in T'. However, we have just built infinitely many distinct functions all of which will pass through any point t' we choose from T_-. How is this fact consistent with the Uniqueness Theorem? Have we done something wrong? Think about it – carefully re-reading the text of the theorem if you are stuck – and then check the next paragraph to see if you are correct.

  Answer: The Uniqueness Theorem applies to *a neighborhood of the point it is given*: it says that its two given solutions are identical **near that point** but says nothing about what happens further on. There is no contradiction because every negative number has a bit of "breathing room" between itself and the $t = 0$ axis, where the solutions diverge.

2 Bifurcation Values

2.1 Drawing a bifurcation diagram

The diagram below corresponds to $\frac{dy}{dt} = (y-1)(y^2 - \mu^2)$.



Steps

1. Draw a μy-plane.
2. Draw curves where $dy/dt = 0$. I have drawn those curves in red above; they are $y = 1$, $y = \mu$, and $y = -\mu$. (Graph the curves as if the plane were the ordinary xy-plane, with μ substituted for x in the function names).
3. Use sign analysis to shade in regions by sign of dy/dt (suggested legend: shade = negative, no shade = positive)
   (a) In particular, what sign does dy/dt take when y is very large and positive? very large and negative? *etc.*
   (b) Shade in the other regions by substituting estimated values and visualizing polynomials for fixed values of μ. For example, in the bounded triangular region containing $(0, \epsilon)$ for small ϵ can be shaded because $(\epsilon - 1)$ will be negative and $\epsilon^2 - \mu^2$ will be positive. The region lower on the y-axis can also be shaded, either by the fact that dy/dt decreases w/o bound as y does or the fact that $y = 0$ is a double root of $(y-1)y^2$ (and therefore does not cause the polynomial to change sign as y decreases).

4. Find bifurcation values based on changes in the nature of phase lines (cross sections).

   The diagram reveals that our bifurcation points are $\boxed{\mu = 0, \pm 1}$.

3 First-order linear non-homogeneous equations

3.1 FAQ

- *Will I be graded on my ability to derive the integrating factor?*

  No.

- *Should I know – by heart – how to derive the integrating factor?*

  Yes.

- *When should I use the integrating factor instead of lucky guess? And v.v.?*

  You may use lucky guess ONLY IF **both** of the following are true:

  1. the coefficients of your DE are constants, and
  2. your non-homogeneous part ($b(t)$) is a polynomial, an exponential, a sine/cosine, or a sum of these three functions.

  It's usually more efficient and less error-prone to use lucky guess for those DEs to which it applies. However, you may use any legal method to solve a problem if none is prescribed by the exam, and for DEs with easily-integrable $b(t)$ using both may be a good way of checking your answer if you finish with spare time.

3.2 Lucky guess method

Suppose we are given the following differential equation:

$$L[y] = f(t) \tag{3}$$

where $L[y]$ is just shorthand for the function of derivatives of y on the LHS and $f(t)$ is a non-zero function of t. Then the general solution to this equation is

$$\boxed{y = y_h + y_p}$$

where y_h is the general solution to the *homogeneous equation*

$$L[y_h] = 0$$

and y_p is ONE particular solution to (3).

Example

$$L[y] = y' - y = t$$

The homogeneous equation is

$$y' - y = 0$$

whose solution is

$$y_h = c \cdot e^t$$

Our lucky guess for a particular solution is

$$y_p = At + B$$

We take the derivative and substitute into the LHS

$$\begin{aligned} (y_p)' &= A \\ t = L[y_p] &= (y_p)' - y_p \\ &= A - (At + B) \\ &= -At + (A - B) \end{aligned}$$

and finally compare coefficients:

$$\begin{gathered} = 1 \cdot t + 0 \\ \boxed{A = -1} \\ \boxed{B = -1} \end{gathered}$$

Thus we obtain $\boxed{y_p = -t - 1}$, and therefore a general solution of

$$\boxed{y(t) = c \cdot e^t - (t + 1)}$$

3.3 One other note

If a lucky guess y_p already solves the homogeneous equation 3.2 then it cannot satisfy the DE for any $f(t)$: this is because, by definition,

$$L[y_p] = 0 \neq f(t)$$

In such a situation, multiply your lucky guess by t; for example, if our original DE had been

$$y' - y = e^t$$

then our lucky guess would have been

$$y_p = Ate^t$$

resulting in a derivative of

$$(y_p)' = A(t + 1)e^t$$

a particular solution

$$y_p = te^t$$

and the general solution

$$y(t) = (t + c)e^t$$