Name (please PRINT):

Out of fairness to your classmates, please stay on this page and <u>DO NOT BEGIN</u> until told otherwise.

Sloppy handwriting increases the chance of grading errors: please write from TOP TO BOTTOM, moving columns of work from LEFT TO RIGHT with STRAIGHT MARGINS in between. Ensure that no work is overlooked by clearly marking any point at which you make an exception to these guidelines.

- Close bags and silence electronics – this quiz is closed-resource.
- If you are still working when time is called, you must stop immediately and bring your quiz to the front. **<u>Absolutely no writing</u>** after time is called.
- Write your printed name on all sheets containing work.
- Box your final answers.
- As much as possible, rubrics are designed so that your grade will not "cascade" from a mistake early in a problem: move on if you have trouble for too long in an early subproblem.
- While you generally need not write in short essay form, you must demonstrate knowledge of course material, supplementing your mathematical notation with words if necessary. In particular, you must
  - *explicitly* cite any theorems you use from the course and
  - write conclusions using at least a few words.

Quiz 5

Score: (curved, out of 20)

NOTE. Questions 1 and 2a on this quiz are multiple choice. No partial credit will be awarded but, on the other hand, there will be no SAT-style penalties for guesses or wrong answers. You should thus attempt to answer every question.

Problem 1 [8 points]

Classify the following four statements as T (true) or F (false). When a matrix A is referred to, assume it to be a 2×2 matrix with real entries and non-zero determinant. Note that true means *true in general with no further conditions.*

1. ________ Any two eigenvectors of an eigenvalue λ of a matrix A must be linearly dependent.
   False, because a repeated eigenvalue may have two linearly independent eigenvectors.

2. ________ If $L[y]$ signifies a second order linear operator,[1] then the equation
   $$L[y] = \tan t \tag{1}$$
   has a solution of the form
   $$y = c \cdot \tan t \tag{2}$$
   where c is a constant.

   False, because the Method of Undetermined Coefficients cannot be used with non-homog. parts of the form $\tan t$.

[1] *i.e.,* the left-hand side of a linear differential equation of second order

3. ________ When all eigenvalues of A are real, the origin, as an equilibrium point of the equation

$$\vec{x}' = A\vec{x} \tag{3}$$

is either unstable or asymptotically stable.
True.

4. ________ Let a, b, c, σ, and τ be real constants. Suppose we are solving the equation

$$ay'' + by' + cy = \sin(\tau t) \tag{4}$$

If the equation

$$ar^2 + br + c = 0 \tag{5}$$

has solutions $r = \sigma \pm i\tau$, then by the Method of Undetermined Coefficients it is correct to guess

$$y_p = t\,(a\cos(\tau t) + b\sin(\tau t)) \tag{6}$$

as a particular solution.
False. We only multiply by t when y_p in its current form is *already a solution of the homogeneous equation.* So we multiply by t in the above equation if $\sigma = 0$ as well, but NOT in general.

You may use this page for work.

Problem 2a [8 pts]

Each of the four lettered phase portraits shown on pages 6 and 7 corresponds to the equation

$$\vec{x}' = \mathbf{A}_i \vec{x}$$

for one of the following five matrices:

$$\mathbf{A}_1 = \begin{pmatrix} -1 & 5 \\ -1 & 1 \end{pmatrix}; \quad \mathbf{A}_2 = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}; \quad \mathbf{A}_3 = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix};$$

$$\mathbf{A}_4 = \begin{pmatrix} 3 & -1 \\ 4 & -2 \end{pmatrix}; \quad \mathbf{A}_5 = \begin{pmatrix} 1 & -1 \\ 2 & -1 \end{pmatrix}$$

In the below table, match each phase portrait to a matrix. Each portrait corresponds to *exactly one* matrix in the list.

(Example: write 5 next to A if phase portrait A is the phase portrait for the equation $\vec{x}' = \mathbf{A}_5 \vec{x}$.)

Portrait	Matrix No.
A	2
B	3
C	4
D	1

In all figures, arrows point in the direction of increasing t and black curves represent solutions.

Figure 1: **Portrait A**

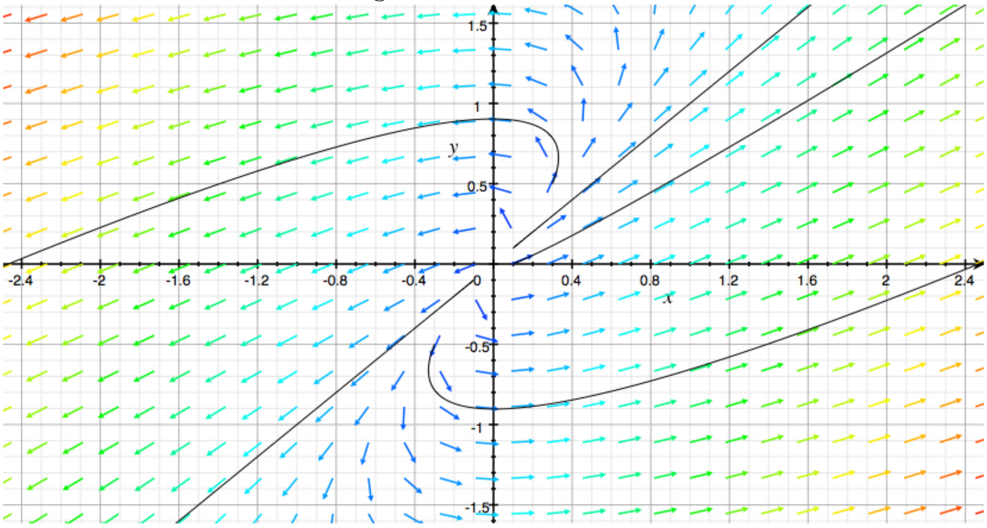


Figure 2: **Portrait B**

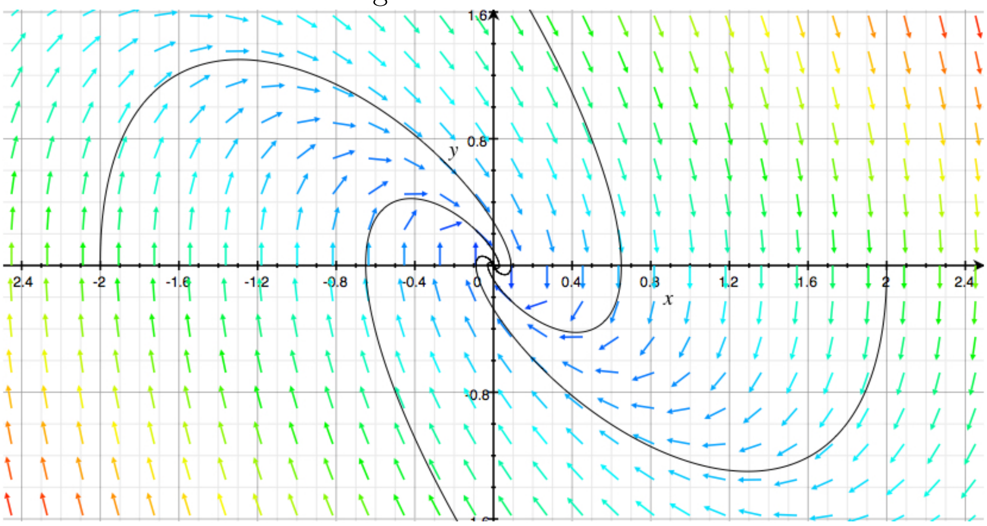


Figure 3: **Portrait C**

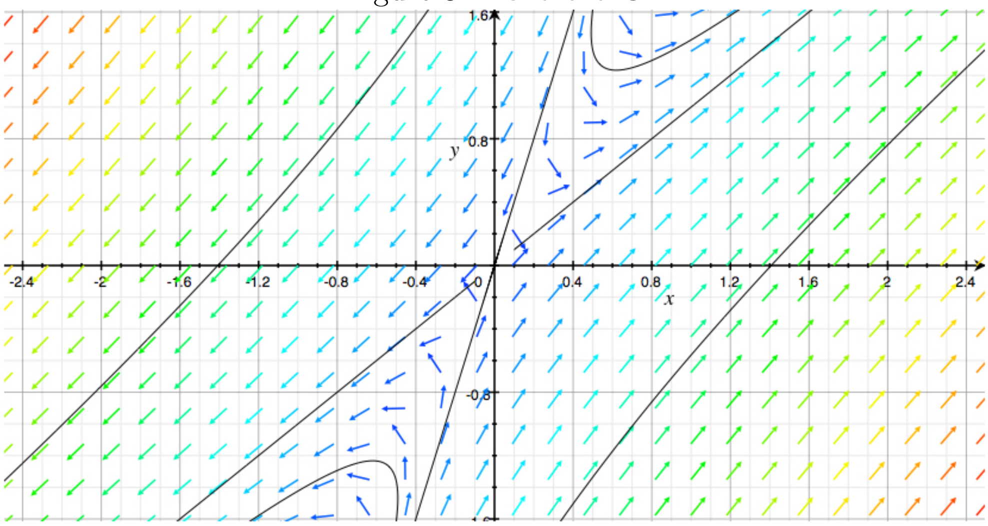
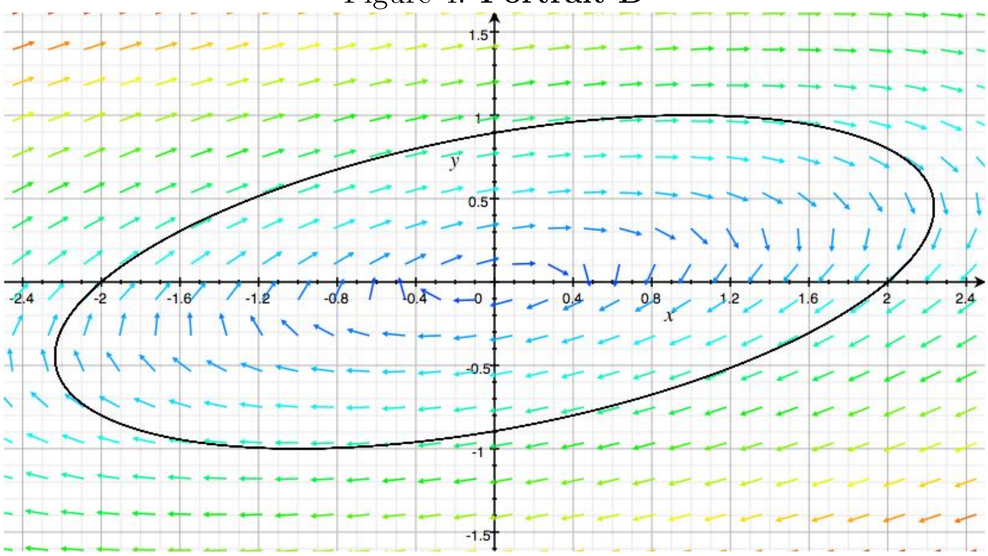


Figure 4: **Portrait D**



Problem 2b [4 pts]

Draw a phase portrait for the equation corresponding to the unmatched matrix in Problem 2a. You do not need to calculate a full solution; an approximate drawing is all right. However, do include arrows in the direction of increasing t and all solutions lying along eigenvector lines (if any).

HINT: There aren't any.

Answer The unmatched matrix is $\mathbf{A}_5$. We start by finding its eigenvalue-vector pairs; its eigenvalues are the solutions to

$$\begin{aligned} 0 = det(A - \lambda I) &= \begin{vmatrix} 1-\lambda & -1 \\ 2 & -1-\lambda \end{vmatrix} && (7) \\ &= (1-\lambda)(-1-\lambda) - (-1)(2) && (8) \\ &= -1 + \lambda^2 + 2 && (9) \\ &= \lambda^2 + 1 && (10) \end{aligned}$$

which are $\pm i$. So we have a center at the origin. To determine the direction of the arrows, simply multiply a vector along the solution by $\mathbf{A}_5$; in this case, we have $dx/dt = 1$ and $dy/dt = 2$ at $(1, 0)$, indicating that that the arrows should point anti-clockwise. (You can distinguish between this portrait and Portrait D in this way.)

You may use this page for work.