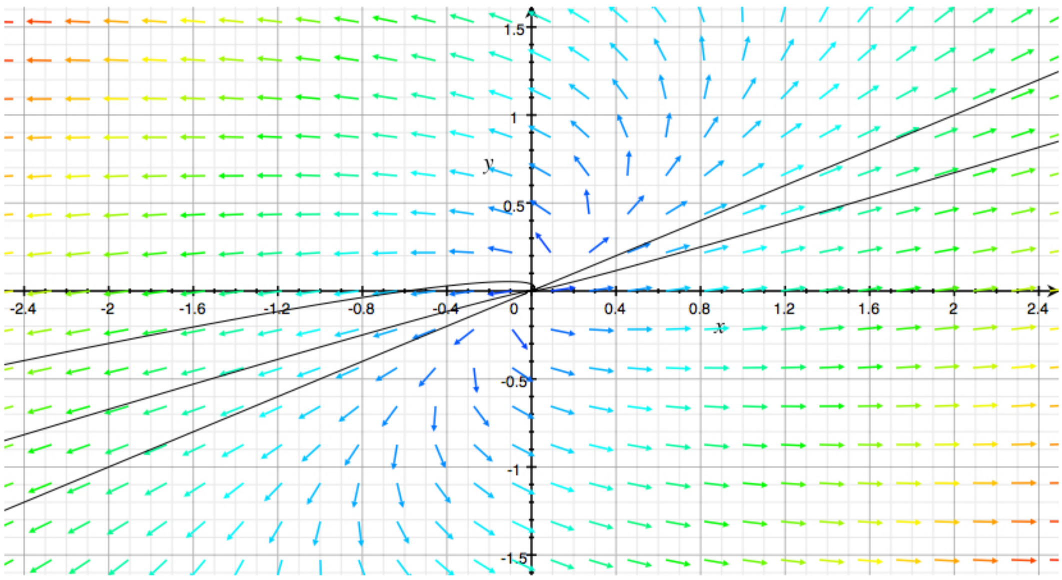
Problem 1 [10 pts]

Consider the matrix equation

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} 6 & -4 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{1}$$

(a) Sketch a phase portrait of solutions to (1). A fully correct solution will, in particular, indicate the orientation of solutions as $t \to \infty$ and include all straight-line solutions, if any.



(b) Solve the equation if $x = 1$ and $y = 0$ at time $t = 0$.

Answer

$$0 = det(A - \lambda I) = \begin{vmatrix} 6-\lambda & -4 \\ 1 & 2-\lambda \end{vmatrix} \tag{2}$$

$$= (6-\lambda)(2-\lambda) + 4 \tag{3}$$

$$= 16 - 8\lambda + \lambda^2 \tag{4}$$

$$= (\lambda - 4)^2 \tag{5}$$

So the eigenvalue of A is 4, yielding a single linearly independent eigenvector

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{0} \tag{6}$$

and a generalized eigenvector

$$\begin{bmatrix} 2 & -4 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \tag{7}$$

whence the general solution

$$Y(t) = e^{4t} \left(c_1 \left(t \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) \tag{8}$$

At time $t = 0$,

$$Y(0) = \left(c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{9}$$

So $c_1 = 1$ and $c_2 = 0$, yielding

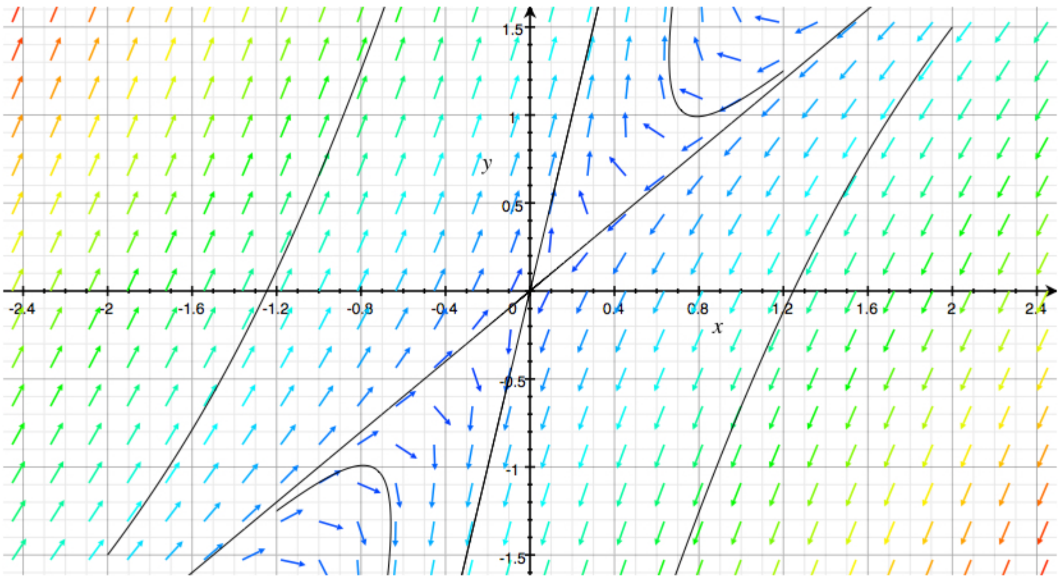
$$\boxed{Y(t) = e^{4t} \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right)}$$

Problem 2 [10 pts]

Consider the matrix equation

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ -5 & 4 \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} \tag{10}$$

(a) Sketch a phase portrait of solutions to (2). A fully correct solution will, in particular, indicate the orientation of solutions as $t \to \infty$ and include all straight-line solutions, if any.



(b) Solve the equation if $x = 1$ and $y = 3$ at time $t = 0$.

Answer

$$0 = det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} \tag{11}$$

$$= (-2-\lambda)(4-\lambda) + 5 \tag{12}$$

$$= -3 - 2\lambda + \lambda^2 \tag{13}$$

$$= (\lambda - 3)(\lambda + 1) \tag{14}$$

So the eigenvalues of A are -1 and 3, from which we find the corresponding eigenvectors

$$\begin{bmatrix} -2-(-1) & 1 \\ -5 & 4-(-1) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -5 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{0} \tag{15}$$

$$\begin{bmatrix} -2-3 & 1 \\ -5 & 4-3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 & 1 \\ -5 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \vec{0} \tag{16}$$

whence the general solution

$$Y(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \tag{17}$$

At time $t = 0$,

$$Y(0) = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \tag{18}$$

So $c_1 = c_2 = 1/2$, yielding

$$\boxed{Y(t) = \frac{1}{2} \left(e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + e^{3t} \begin{bmatrix} 1 \\ 5 \end{bmatrix} \right)}$$