1 Question 1

Consider the differential equation

$$3y^3 \cdot \frac{dy}{dx} - 2xy = 0; \quad y(3) = 2 \tag{1}$$

(a) [3 pts] Show that equation (1) is NOT an exact differential equation. If you use a theorem, be sure to confirm *all* its hypotheses[1] and be precise with your wording.

Answer We have

$$M(x,y) = -2xy \tag{2}$$
$$N(x,y) = 3y^3 \tag{3}$$

and thus

$$\frac{\delta M}{\delta y} = -2x \tag{4}$$
$$\frac{\delta N}{\delta x} = 0 \tag{5}$$

All four of these functions are continuous everywhere (so in any rectangle we choose), but $M_y \neq N_x$ in any rectangle; therefore, the DE is not exact.

(b) [7 pts] Solve (1) using methods of exact differential equations. You will need to use an integrating factor, and the easiest one is a function of y only.

Answer We need to find μ such that μM, μN, $(\mu M)_y$, and $\mu N)_x$ are all continuous in a rectangle around $(3, 2)$ and $(\mu M)_y = (\mu N)_x$. We start with

[1] Hypotheses are the conditions that need to be true for a theorem to apply; e.g., a triangle being right to use the Pythagorean Theorem

the latter requirement:

$$
\begin{aligned}
(\mu M)_y &= (\mu N)_x && (6)\\
\mu_y M + \mu M_y &= \mu_x N + \mu N_x && (7)\\
\mu_y M + \mu(M_y - N_x) &= 0 && (8)\\
\mu_y + \mu \frac{M_y - N_x}{M} &= 0 && (9)\\
\mu_y + \frac{1}{y}\mu &= 0 && (10)
\end{aligned}
$$

Since μ is a one-variable function, this last equation is a first-order linear DE in y, with solution

$$\mu(y) = \frac{1}{y} \tag{11}$$

where the absolute value can be dropped because we are working close to $y = 2$. The resulting DE is

$$-2x + 3y^2 \cdot \frac{dy}{dx} = 0 \tag{12}$$

whose M and N are polynomials in x and y and are therefore continuous everywhere; hence, by design, it is also exact.

$$
\begin{aligned}
\psi(x, y) &= h(y) + \int M\, dx && (13)\\
&= h(y) \int -2x\, dx && (14)\\
&= h(y) - x^2 && (15)\\
\frac{\delta \psi}{\delta y} &= N(x, y) && (16)\\
&= 3y^2 && (17)\\
&= h'(y) && (18)
\end{aligned}
$$

Thus $h(y) = y^3$ and $\psi(x, y) = y^3 - x^2$, yielding a general solution of

$$y^3 - x^2 = C \tag{19}$$

Substituting initial conditions yields

$$2^3 - 3^2 = 8 - 9 = -1 \tag{20}$$

$$\boxed{x^2 - y^3 = 1} \tag{21}$$

2 Question 2

Consider the differential equation

$$y' = 1 - y^4 \tag{22}$$

(a) [1 pt] Find all equilibrium points of (22).

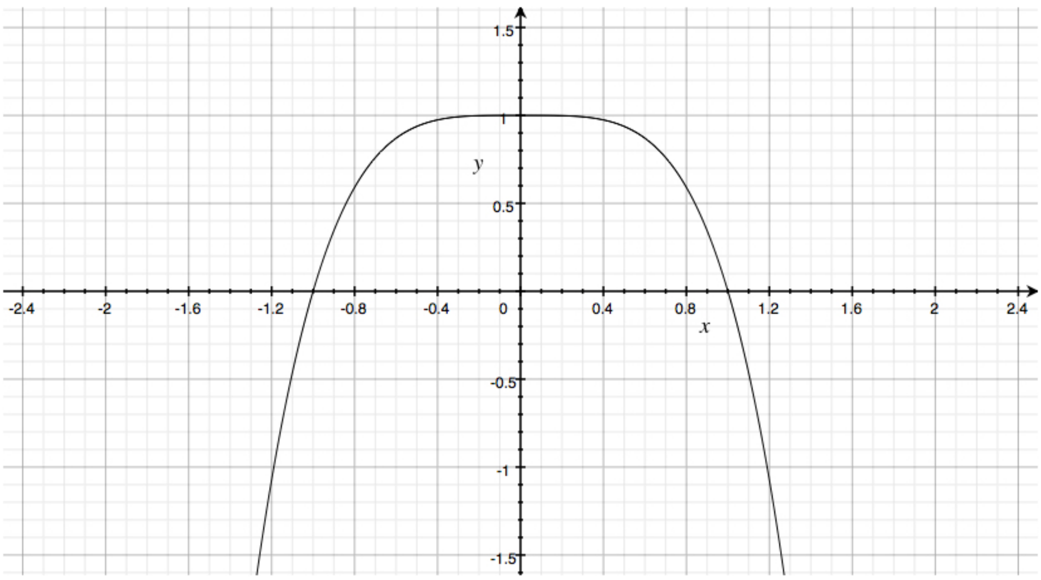
Answer

$$y' = 0 = 1 - y^4 = (1 - y^2)(1 + y^2) = (1 - y)(1 + y)(1 + y^2)$$

So the equilibrium points are at $\boxed{y = \pm 1}$.

(b) [1 pt] Draw a graph of $f(y)$ vs. y.

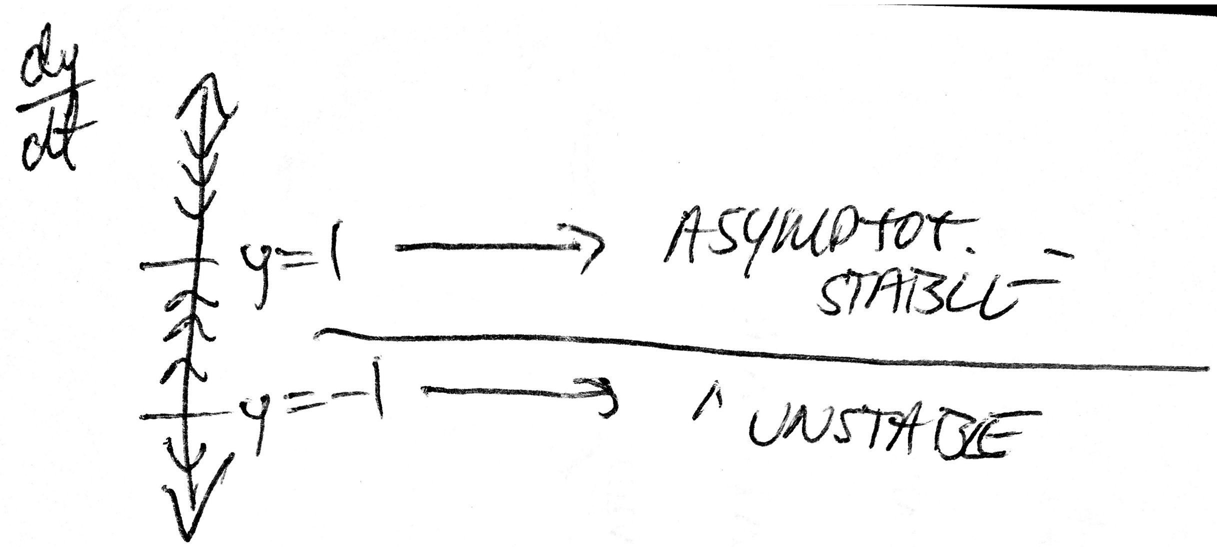
Answer Should look more or less like this:



Essentials: roots at ± 1, vertex at $(0, 1)$, concave down, symmetric w/r to vertical axis.

(c) [2 pts] Use this graph to draw a phase line, then classify the equilibrium points.

Answer



(d) [6 pts] On a single ty-plane, draw reasonably accurate solutions to (22) with the following initial conditions:

- $y(0) = 1.25$
- $y(0) = 1$
- $y(0) = 0.75$
- $y(0) = -0.75$
- $y(0) = -1$
- $y(0) = -1.25$

Your graph should be correct up to end behavior, the positions of solution curves relative to one another, and behavior near equilibrium solutions.

Answer