Quiz 2x Solutions

Question 1

Solve the initial value problem

$$y''(t) + 4y'(t) + 8y(t) = 0; \quad y(\pi/2) = e^{-\pi},\ y'(\pi/2) = -4e^{-\pi} \tag{1}$$

Answer The characteristic polynomial is x^2+4x+8, which has discriminant $4^2 - 4*1*8 = 16 - 32 < 0$. We can find roots by completing the square:

$$\begin{aligned} x^2 + 4x + 8 &= 0 \\ x^2 + 4x + 4 &= -4 \\ (x+2)^2 &= -4 \\ x + 2 &= \pm 2i \\ x &= -2 \pm 2i \end{aligned}$$

So the general solution is

$$y(t) = e^{-2t}\big(A\cos(2t) + B\sin(2t)\big)$$

With the first initial condition, we obtain

$$e^{-\pi} = y(\pi/2) = Ae^{-\pi}\cos(\pi) = -Ae^{-\pi}$$

whence $A = -1$. This fact gives us

$$\begin{aligned} y(t) &= e^{-2t}\big(B\sin(2t) - \cos(2t)\big) \\ y'(t) &= e^{-2t}\big(2B\cos(2t) + 2\sin(2t) - 2B\sin(2t) + 2\cos(2t)\big) \\ -4e^{-\pi} = y'(\pi/2) &= e^{-\pi}\big(-2B - 2\big) \\ B &= 1 \end{aligned}$$

Hence

$$\boxed{y(t) = e^{-2t}\big(\sin(2t) - \cos(2t)\big)}$$

Question 2

Using only methods from Chapters 2 and 3, show that the solutions to the DE

$$x^2y'' - 3xy' + 3y = 0 \tag{2}$$

are exactly the functions

$$y = c_1x^3 + c_2x \tag{3}$$

where c_1 and c_2 are arbitrary constants (*i.e.,* show both that these functions solve the DE and that they are the only solutions).

Answer Several correct methods exist, among which:

Call $y_1 := x^3$ and $y_2 := x$. Then

$$\begin{aligned} y_1 &= x^3 \\ y_1' &= 3x^2 \\ y_1'' &= 6x \end{aligned}$$

and

$$\begin{aligned} y_2 &= x \\ y_2' &= 1 \\ y_2'' &= 0 \end{aligned}$$

Substituting,

$$\begin{aligned} x^2(6x) - 3x(3x^2) + 3(x^3) &= 6x^3 - 9x^3 + 3x^3 \\ &= 0 \\ x^2(0) - 3x(1) + 3(x) &= 0 - 3x + 3x \\ &= 0 \end{aligned}$$

So both y_1 and y_2 are solutions to the DE [**2 pts**]. Since the DE is linear homogeneous, any linear combination of y_1 and y_2 is also a solution by the Principle of Superposition. Furthermore, the Wronskian of x^3 and x is

$$(x^3) \cdot 1 - (3x^2) \cdot x = -2x^3$$

which is not identically zero; hence, linear combinations of x and x^3 form the only solutions.