Quiz 1 SOLUTIONS

Question 1

Give the order of the following DEs (1 pt each) and decide if they are linear or non-linear (1 pt each). No justification is required.

(A) $y'' = ty + 1$ **second-order linear**

(B) $(y')^2 = ty + 1$ **first-order non-linear**

(C) $t^2y' - y = \sin t$ **first-order linear**

(D) $y'y = 1$**first-order non-linear**

Question 2

Two people of the same age, P and Q, wish to save for retirement at some point. Their investments will earn 5% continuous interest, neither will begin with any capital at the time they start saving, and P (resp., Q) will save at the *continuous* rate of a (resp., b) dollars per year.

(A) (2 pts) If P (resp., Q) represents the size of P's (resp., Q's) portfolio at time t, write a DE

- for P in terms of P and a, and
- for Q in terms of Q and b

Answer

$$\frac{dP}{dt} = 0.05P + a \tag{1}$$

$$\frac{dQ}{dt} = 0.05Q + b \tag{2}$$

(B) (2 pts) Let $t = 0$ signify the time at which P and Q turn 25.

Whereas P plans to start saving immediately at 25, Q decides to prioritize other things and wait until age 30. Write initial conditions representing these data.

Answer

$$P(0) = 0 \tag{3}$$
$$Q(5) = 0 \tag{4}$$

Remember that $t = 5$ is the moment five years ahead of the time when P and Q turn 25, hence the moment when Q turns 30. At the time they begin saving – and *only* at that time, since the accounts accrue interests and contributions – neither has any money.

(C) (4 pts) You have two initial value problems, one for P and one for Q. Solve both to find $P(t)$ and $Q(t)$.

Answer Several correct ways exist. The equation for P is first-order linear with standard form

$$P' - 0.05P = a$$

yielding $\mu(t) = e^{-t/20}$ and therefore

$$P(t) = \frac{1}{\mu(t)} \int \mu(s) g(s)\, ds \tag{5}$$
$$= e^{t/20} \int e^{-s/20} a\, ds \tag{6}$$
$$= a e^{t/20} \big[-20 e^{-t/20} + A \big] \tag{7}$$
$$= -20a + A e^{t/20} \tag{8}$$

Analogously,

$$Q(t) = -20b + B e^{t/20} \tag{9}$$

We now substitute the initial values

$$P(0) = 0 = -20a + A e^{0/20} \tag{10}$$
$$A = 20a \tag{11}$$
$$Q(5) = 0 = -20b + B e^{5/20} \tag{12}$$
$$B = 20b e^{-1/4} \tag{13}$$

to obtain

$$P(t) = 20a(e^{t/20} - 1) \tag{14}$$
$$Q(t) = 20b(e^{(t-5)/20} - 1) \tag{15}$$

(D) (4 pts) Both P and Q wish to have \$1 million saved by age 65. Find b/a, the proportion of the amount Q must save annually to the amount P must save annually. Using the approximations

$$e^2 - 1 \approx 26/4 \tag{16}$$
$$e^{1.75} - 1 \approx 19/4 \tag{17}$$

estimate what percentage more than P's annual payment Q's annual payment must be for Q having delayed saving by five years. (An answer within five percentage points will be correct if supported by correct work.)

Answer Set $t = 40$ and $P(40) = 10^6$, then solve for a and b:

$$P(40) = 10^6 = 20a(e^{40/20} - 1) \tag{18}$$
$$Q(40) = 10^6 = 20b(e^{(35)/20} - 1) \tag{19}$$
$$a = \frac{5 \cdot 10^4}{e^2 - 1} \tag{20}$$
$$b = \frac{5 \cdot 10^4}{e^{1.75} - 1} \tag{21}$$

So

$$b/a = \frac{e^2 - 1}{e^{1.75} - 1} \tag{22}$$
$$\approx 26/19 \tag{23}$$
$$\approx 27/20 \tag{24}$$

So Q will need to pay about 35% more per month. (The exact percentage is 34.3762% up to four decimal places.)