244 Review, Day 2 (no hints)

Question IV

Use Euler's formula and properties of exponents to prove the double angle identities from trigonometry:

- $\sin(2\theta) = 2\sin\theta\cos\theta$

- $\cos(2\theta) = \cos^2\theta - \sin^2\theta$

Question V

Without using the Method of Variation of Parameters, find a general solution to the differential equation

$$y'' - 2y' + y = (t-3)(t-1) + \cos(t)\cos(2t)$$

(See HINTS 1-2, next page.)

Question VI [revised]

Suppose that the population of Species X in hundreds is represented by $x(t)$ at time t and that of Species Y in hundreds is represented by $y(t)$ at time t, with

$$x'(t) = x(3 - x - 2y)$$
$$y'(t) = y(1 - y - x)$$

Find all possible limits of the populations of Species X and Y as $t \to \infty$, assuming only that both populations are strictly positive at time zero. (A fully correct answer will appeal to an existence and uniqueness theorem and rigorously explain why, mathematically speaking, the two populations cannot grow without bound.) (See HINTS 3-4, next page.)

Hints

HINT 1: There is no direct, obvious application of any method from the course (other than perhaps MOVOP) to the right-hand side as written.

HINT 2: Recall the identity used to write solutions to spring-mass systems in the form
$$R\cos(\omega t - \delta)$$

HINT 3: Chart the x- and y-nullclines of this system on a set of axes. In what region is $dx/dt < 0$ satisfied? And in what region is $dy/dt < 0$? Where do these regions overlap? Now draw a parametric curve $(x(t), y(t))$ satisfying the system such that $x(t) \to \infty$ and $y(t) \to \infty$. What contradiction becomes apparent?

HINT 4: You should obtain four equilibrium points, of which only **one** is a possible population limit. It should be straightforward to explain why two of the four are impossible. Of the other two, at least one will be on the axes, and you should carefully explain – using an existence/uniqueness theorem – why no two non-zero populations growing according to the system can end with its corresponding values.