Complexification

Complexification is a strategy for calculating integrals and applying the Method of Undetermined Coefficients. It relies on **Euler's Formula**, namely

$$e^{it} = \cos t + i \sin t \tag{1}$$

and the observation that if

$$a + bi = c + di \tag{2}$$

with a, b, c, d real, then $a = c$ and $b = d$; *i.e.,* we can match real and imaginary parts of complex numbers.

1 Integrals

A good place to start is by calculating

$$\int \cos(3t)\, dt \tag{3}$$

Note that

$$e^{i3t} = \cos(3t) + i \sin(3t) \tag{4}$$

and so

$$\int e^{i3t}\, dt = \int \cos(3t)\, dt + i \int \sin(3t)\, dt \tag{5}$$

Thus

$$\int \cos(3t)\, dt = Re(\int e^{i3t}\, dt) \tag{6}$$

where Re signifies the real part. In this section we are going to omit arbitrary constants for brevity and let equality signify *equality up to an arbitrary scalar constant.*

Integrals and derivatives of exponential functions follow the same rules with complex arguments as with real ones, so

$$\int e^{i3t}\,dt = \frac{1}{3i}e^{i3t} \tag{7}$$

$$= \frac{1}{3i}\big(\cos(3t) + i\sin(3t)\big) \tag{8}$$

$$= \frac{1}{3}\sin(3t) + \frac{1}{3i}\cos(3t) \tag{9}$$

$$= \frac{1}{3}\sin(3t) - \frac{i}{3}\cos(3t) \tag{10}$$

So

$$\int \cos(3t)\,dt = \frac{1}{3}\sin(3t) \tag{11}$$

which is what we would obtain from taking the integral using Cal I/II methods.

Now consider a more complicated example, also possible – though tedious – with first-year calculus methods:

$$\int t\sin^2 t\,dt \tag{12}$$

The laws for adding and multiplying exponents continue to apply to complex-valued exponents, a fact that, incidentally, allows us to prove two trig identities:

$$e^{i2t} = (e^{it})^2 \tag{13}$$

$$\cos(2t) + i\sin(2t) = \big(\cos t + i\sin t\big)^2 \tag{14}$$

$$= \big(\cos^2 t - \sin^2 t\big) + i\big(2\sin t\cos t\big) \tag{15}$$

$$\therefore \cos(2t) = \cos^2 t - \sin^2 t \tag{16}$$

$$\therefore \sin(2t) = 2\sin t\cos t \tag{17}$$

We use one more identity to obtain

$$\cos(2t) = (1 - \sin^2 t) - \sin^2 t \tag{18}$$

$$= 1 - 2\sin^2 t \tag{19}$$

So

$$\int t\big(1 - 2\sin^2 t\big)\, dt = \int t\cos(2t)\, dt \tag{20}$$

$$= Re\Big(\int te^{i2t}\, dt\Big) \tag{21}$$

It takes only one step of integration by parts to show that

$$\int te^{i2t}\, dt = \frac{e^{i2t}}{4}\big(1 - 2it\big) \tag{22}$$

$$= \frac{1}{4}\big(\cos(2t) + i\sin(2t)\big)(1 - 2it) \tag{23}$$

$$= \frac{1}{4}\Big(\cos(2t) + 2t\sin(2t) + i\big(\sin(2t) - 2t\cos(2t)\big)\Big) \tag{24}$$

$$\int t\big(1 - 2\sin^2 t\big)\, dt = \frac{1}{4}\big(\cos(2t) + 2t\sin(2t)\big) \tag{25}$$

$$\frac{t^2}{2} - \frac{1}{4}\big(\cos(2t) + 2t\sin(2t)\big) = 2\int t\sin^2 t\, dt \tag{26}$$

yielding

$$\boxed{\int t\sin^2 t\, dt = \frac{t^2}{4} - \frac{1}{8}\big(\cos(2t) + 2t\sin(2t)\big)} \tag{27}$$

Note that step (26) treats the value of an indefinite integral like a variable: this step is valid because equality up to a constant is preserved by addition and multiplication of scalars (CHALLENGE: would the same be true for addition/multiplication of *functions*?)

In general, if $f(t)$ is a real function then

$$\int f(t)\cos(bt) = Re\big(\int f(t)e^{ibt}\, dt\big) \tag{28}$$

$$\int f(t)\sin(bt) = Im\big(\int f(t)e^{ibt}\, dt\big) \tag{29}$$

$$\tag{30}$$

where Im signifies the imaginary part.

2 Method of Undetermined Coefficients

When solving a linear DE of the form

$$L[y] = g(t)$$

where $g(t)$ is the product of a polynomial $p(t)$ of degree n (possibly 0), an exponential e^{at} (where a is possibly 0), and a $\sin(bt)$ or $\cos(bt)$, it can be easier to complexify for purposes of finding a particular solution. There are four steps:

1. Replace $g(t)$ with $p(t)e^{(a+bi)t}$
2. Find $\tilde{y}$ of the form $c \cdot P(t)e^{(a+bi)t}$, where c is complex and $P(t)$ is t^s times a polynomial of degree n (s the least non-negative integer such that $\tilde{y}$ is not a solution to the homogeneous version of the equation)
3. Apply Euler's formula and FOIL $\tilde{y}$ into real and imaginary parts.
4. Set y_p to be the real (cos) or imaginary (sin) part.

Example: 3.5 #18

Find y_p for

$$L[y] = y'' + 2y' + 5y = 4e^{-t}\cos(2t)$$

1. Replace g(t):
$$y'' + 2y' + 5y = \mathbf{4e^{(-1+2i)t}}$$
2. Setting $z = -1 + 2i$, find $\tilde{y} = cte^{zt}$ ($s = 1$):

$$\begin{aligned}
5\tilde{y} &= 5cte^{zt} \\
2\tilde{y}' &= 2c(zt+1)e^{zt} \\
\tilde{y}'' &= c(z^2t + 2z)e^{zt} \\
\hline
L[\tilde{y}] &= ce^{zt}\big((z^2+2z+5)t + (2+2z)\big) \\
&= c(2+2z)e^{zt} = 4e^{zt} \\
\therefore c(2+2z) &= 4 \\
\therefore c &= -i
\end{aligned}$$

3. Apply Euler and FOIL:

$$\begin{aligned}\tilde{y} &= cte^{zt} && (31)\\ &= -ie^{-t}t\big(\cos(2t) + i\sin(2t)\big) && (32)\\ &= e^{-t}t\big(\sin(2t) - i\cos(2t)\big) && (33)\end{aligned}$$

4. Conclude that

$$\boxed{y_p = te^{-t}\sin 2t}$$