Review for Exam 1

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NOTE. This document is intended as a supplement for Exam 1 study, NOT as an exhaustive list of topics that will be covered, nor as a substitute for other materials.

1 Chapter 1

1. What is the value of $\log(1)$?
2. The population, P, of a certain city starts at ten thousand at $t = 0$, grows at a rate directly proportional to itself, and reaches thirty thousand within fifteen years ($t = 15$). Write a differential equation modelling this situation and solve it (for readability's sake, P can be given in thousands or ten thousands).
3. Indicate whether the following DEs are *linear*, *homogeneous*, or *separable* (any number of descriptors, including zero, may apply to each DE)
   - $\frac{dy}{dt} = 1$
   - $\frac{dy}{dt} = ty$
   - $\frac{dy}{dt} = e^{ty}$
   - $\frac{dy}{dt} + y = 0$
   - $\frac{dy}{dt} + y^2 = 0$
4. Write a basic predator-prey model (use any letters but indicate which is predator and which is prey). What should the signs of the coefficients be, and why?
5. Write the basic form of DE corresponding to a logistic model when both of the following are true:
   - The habitat can support at most 2000 individuals.
   - The variable t is in years, and the population will be increasing at a rate of 100 individuals per year at the instant that the population reaches 1000 individuals.

6. What two things are wrong with the following solution of the differential equation $\frac{dy}{dt} = 2y$?

$$\begin{aligned} \frac{dy}{dt} &= 2y \\ \frac{dy}{y} &= 2\,dt \\ \log(y) &= 2t \\ \therefore y &= Ce^{2t} \end{aligned}$$

(the solution happens to be technically correct but the work omits two important considerations)

7. Find an approximation for $x(4)$ when

$$\frac{dx}{dt} = tx$$

and $x(0) = 1$, using the Euler method with $\Delta t = 1$.

8. Give a DE for which:
   - both Existence and Uniqueness fail at at least one (t, y)
   - Uniqueness, but not Existence, fails at at least one (t, y)

9. **Memorize** the hypotheses and conclusions for these two theorems!

10. Find the bifurcation point(s) for

$$\frac{dy}{dt} = y^2 + \alpha$$

and draw phase lines for a value of α on each side of each bifurcation point.

11. Give an example of an autonomous DE $\frac{dy}{dt} = f(y)$ for which $f(0) = 0$, $f'(0) = 0$, and the general solution has a sink at $y = 0$.

12. Let

$$y' + p(t)y = 0$$

be a homogeneous differential equation, and let

$$y' + p(t)y = q(t)$$

be non-homogeneous. Prove that, if y_1 and y_2 are solutions to the latter equation, then their difference is a solution to the first. (This means that any solution to the latter has to be a solution to the former plus a particular solution!)

13. Derive a general solution from scratch to the non-homogeneous equation above. You should first decide on what the integrating factor should be, then multiply both sides by it and integrate. How would you modify your general solution to account for an initial condition?

14. I am thinking of a function $y(t)$. If I add $2t$ to $y(t)$, I obtain $y''(t)$, and both y and y' vanish at $t = 0$. What is $y(t)$?

15. A tank contains 1000 mL of distilled water with no dye at $t = 0$. A 4mL mixture, comprising 1mL of synthetic dye and 3mL water, is added to the tank each minute, while 3mL leaks out from a small hole in the bottom each minute. Assuming that the dye is always distributed equally through the solution (i.e., is well-mixed), how much dye is in the tank after one hour?

2 Chapters 2 and 3

NOTE. Although matrix exponentials will NOT be tested on this first exam, I recommend practicing with them anyway, both to prepare for Chapter 3 and to be able to check your work on the exam.

For 1-4, consider the equation

$$\frac{d\mathbf{Y}}{dt} = A\mathbf{Y}$$

with

$$\mathbf{Y}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

and

$$\mathbf{Y}'(t) = \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix}$$

For each matrix A, use Euler's method for systems to approximate $\mathbf{Y}(3)$ with step size $\Delta t = 1$. Use methods from Chapter 2 to solve 1 and 2 and verify your answers with matrix exponentials. For extra practice, solve 3 and 4 using matrix exponentials.

1. $A = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}$, $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
2. $A = \begin{pmatrix} 2 & 1 \\ 0 & -1 \end{pmatrix}$, $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
3. $A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$, $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
4. $A = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}$, $\mathbf{Y}(0) = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$
5. Are the above solutions unique? How could we know that they even *existed*, apart from by solving them?

6. Suppose that

$$(x(t), y(t)) = (e^{2t}, 0)$$
$$(x(t), y(t)) = (e^{2t}, e^{-t})$$

solve a linear system of equations $\mathbf{Y}' = A\mathbf{Y}$. Find a particular solution with $(x(0), y(0)) = (1, 3)$. Is this particular solution the **only** solution with that initial condition? Why or why not?