How do I know where bifurcations occur?

Bifurcations occur where the equilibrium points "change in nature." Such a vague description isn't of much help on its own, so let's explore the idea in the context of a specific example. Consider the autonomous DE

$$\frac{dy}{dt} = y^2 + \alpha$$

What are the equilibrium solutions of this DE, and where does it have sinks and sources? The answer to these questions depends on the value of α.

Case 1: If α is negative then we have two zeros of $y^2+\alpha$, hence two equilibrium solutions, $y = \pm\sqrt{|\alpha|}$. The derivative of $y^2 + \alpha$ is negative at the negative root and positive at the positive, so we have a source and a sink.

Case 2: If α is zero then there is a single equilibrium point at $y = 0$. Since $\frac{dy}{dt}$ is positive both above and below $y = 0$ we have increasing functions on both sides, hence a node (note also that $(y^2 + \alpha)' = 2y = 0$).

Case 3: If α is positive then $y^2 + \alpha$ never vanishes; hence there are no equilibrium points. All solutions strictly increase.

It's clear that $\alpha = 0$ is a bifurcation point; what happened is that the sign of the derivative of $\frac{dy}{dt}$ *with respect to* y at the equilibrium points changed there, from positive/negative to zero.

Going back to the general case, suppose we are given an autonomous DE

$$\frac{dy}{dt} = f(y, \alpha)$$

and f is C^1 (i.e., has a continuous derivative with respect to y and α). If, at a certain equilibrium point y_0 and a certain parameter α, we have that

$$\frac{df}{dy}(y_0, \alpha) > 0$$

then, by continuity, there's a "neighbourhood" of (y_0, α) where $\frac{df}{dy}$ stays positive; in other words, we will not have any sudden changes in the "nature" of the equilibrium solution near (y_0, α). (The same logic applies, with obvious changes in wording, if $>$ is replaced with $<$ above.)

Hence, for autonomous equations, bifurcation points can only occur when both

$$f(y, \alpha) = 0$$

and

$$\frac{df}{dy}(y, \alpha) = 0$$

In our example case, we could have seen that $\alpha = 0$ was a bifurcation point by the analysis below:

$$\begin{aligned} f(y,\alpha) &= y^2 + \alpha = 0 \\ f'(y,\alpha) &= 2y = 0 \end{aligned}$$

Combined, the system implies that $\alpha = 0$ at any bifurcation point; we then confirm that it is one using the case analysis on the previous page.