Quiz 8 Rubric

Question 1 [10 points]

Let

$$A := \begin{bmatrix} 1 & -2 & -1 & 0 \\ 0 & 1 & -1 & -2 \\ 0 & 2 & 1 & 2 \\ 1 & 1 & -1 & 0 \end{bmatrix} \qquad \textbf{(1)}$$

be the matrix that has the four vectors as columns. Since the determinant of A is zero:

$Determinant(A)$

$$0 \qquad \textbf{(2)}$$

the columns of the matrix are linearly dependent; *i.e.*, there exists a linear combination of them that equals zero in which not all the coefficients are zero. You need only rearrange these columns and divide by one of the coefficients to get the needed linear combination.

To take the determinant of A, start with the first column since it has the most zeros. Then

$$det(A) = 1 \cdot det\begin{bmatrix} 1 & -1 & -2 \\ 2 & 1 & 2 \\ 1 & -1 & 0 \end{bmatrix} - 0 + 0 - (1) \cdot det\begin{bmatrix} -2 & -1 & 0 \\ 1 & -1 & -2 \\ 2 & 1 & 2 \end{bmatrix} = 6 - 6 = 0$$

Scoring:

- **2 pts**: correctly taking the determinant (must show at least sum of det's of 3x3 matrices, as above)
- **1 pt:** selecting the first column (*i.e.*, the column with the most zeros)
- **4 pt:** citing equivalence between zero determinant and linear dependence
- **3 pt**: explaining why linear dependence is relevant to the solution

While the above answer is preferred, showing an explicit relationship (*e.g.,* by solving for one through linear combinations) is also acceptable as long as all work is shown.

Question 2a [10 pts]

$$A := \begin{bmatrix} 1 & 2 & 1 \\ a & 0 & 2 \\ 1 & a & -3 \end{bmatrix} \tag{3}$$

The determinant of A is **[4 pts]**

$$a^2 + 4\,a + 4 \tag{4}$$

which is zero exactly when **[1 pt]** $a = -2$. Invertibility of a matrix is equivalent to the matrix having non-zero determinant **[3 pts];** hence **[2 pts]** the matrix fails to be invertible when $a = -2$.

Question 2b [extra credit]

$A := \langle A | IdentityMatrix(3) \rangle$

$$A := \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ a & 0 & 2 & 0 & 1 & 0 \\ 1 & a & -3 & 0 & 0 & 1 \end{bmatrix} \tag{5}$$

(Steps may vary.)

Step 1: Add multiples of the first row to the second and third rows.

$$A1 := \begin{bmatrix} 1 & 0 & 0 \\ -a & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} A$$

$$A1 := \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -2\,a & -a+2 & -a & 1 & 0 \\ 0 & -2+a & -4 & -1 & 0 & 1 \end{bmatrix} \tag{6}$$

Step 2: If $a \neq 0, 2$, then multiply the three rows to a common multiple (scalar row multiplication by zero is not allowed; skip to step *E1* in these cases).

$$A2 := map\left(simplify, \begin{bmatrix} a\cdot(a-2) & 0 & 0 \\ 0 & (a-2) & 0 \\ 0 & 0 & 2\,a \end{bmatrix} A1\right)$$

$$A2 := \begin{bmatrix} a\,(-2+a) & 2\,a\,(-2+a) & a\,(-2+a) & a\,(-2+a) & 0 & 0 \\ 0 & -2\,a\,(-2+a) & -(-2+a)^2 & -a\,(-2+a) & -2+a & 0 \\ 0 & 2\,a\,(-2+a) & -8\,a & -2\,a & 0 & 2\,a \end{bmatrix} \quad \textbf{(7)}$$

Step 3: Divide each row by $(a-2)$, eliminate non-zero entries in row 1 and 3 in column 2, then multiply by *a - 2* in row 3.

$$A3 := map\left(simplify, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & a-2 & a-2 \end{bmatrix} . \begin{bmatrix} \frac{1}{a-2} & 0 & 0 \\ 0 & \frac{1}{a-2} & 0 \\ 0 & 0 & \frac{1}{a-2} \end{bmatrix} . A2\right)$$

$$A3 := \begin{bmatrix} a & 0 & 2 & 0 & 1 & 0 \\ 0 & -2\,a & -a+2 & -a & 1 & 0 \\ 0 & 0 & -(a+2)^2 & -a^2 & -2+a & 2\,a \end{bmatrix} \quad \textbf{(8)}$$

Step 4: Divide the last row by $-(a+2)^2$ (okay because $a \neq -2$), eliminate non-zero multiples in row 1 and 2 in column 3, and factor out $\dfrac{1}{(a+2)^2}$.

$$A4 := map\left(simplify, (a+2)^2 \begin{bmatrix} 1 & 0 & 2(a+2)^{-2} \\ 0 & 1 & -(a-2)\cdot(a+2)^{-2} \\ 0 & 0 & -(a+2)^{-2} \end{bmatrix} A3\right)$$

$$A4 := \begin{bmatrix} a\,(a+2)^2 & 0 & 0 & -2\,a^2 & a^2+6\,a & 4\,a \\ 0 & -2\,a\,(a+2)^2 & 0 & -6\,a^2-4\,a & 8\,a & -2\,a\,(-2+a) \\ 0 & 0 & (a+2)^2 & a^2 & -a+2 & -2\,a \end{bmatrix} \quad \textbf{(9)}$$

Step 5: Divide row 1 and row 2 by *a* and -*2a*, respectively.

$$A5 := map\left(simplify, \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{-1}{2\,a} & 0 \\ 0 & 0 & 1 \end{bmatrix} A4\right)$$

$$A5 := \begin{bmatrix} (a+2)^2 & 0 & 0 & -2\,a & a+6 & 4 \\ 0 & (a+2)^2 & 0 & 3\,a+2 & -4 & -2+a \\ 0 & 0 & (a+2)^2 & a^2 & -a+2 & -2\,a \end{bmatrix} \tag{10}$$

Factoring $\dfrac{1}{(a+2)^2}$ back in, the inverse for $a \neq 0, \pm 2$ is

$$A^{-1} = \frac{1}{(a+2)^2} \begin{bmatrix} -2\,a & a+6 & 4 \\ 3\,a+2 & -4 & -2+a \\ a^2 & -a+2 & -2\,a \end{bmatrix}$$

Step E1: If $a = 0$ then we pick back up with

$B2 := eval(A1, a = 0)$

$$B2 := \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 & 1 & 0 \\ 0 & -2 & -4 & -1 & 0 & 1 \end{bmatrix} \tag{11}$$

$$B3 := \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} B2$$

$$B3 := \begin{bmatrix} 1 & 0 & -3 & 0 & 0 & 1 \\ 0 & -2 & -4 & -1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 1 & 0 \end{bmatrix} \tag{12}$$

$$B4 := \begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & -\frac{1}{2} & -1 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} B3$$

$$B4 := \begin{bmatrix} 1 & 0 & 0 & 0 & \frac{3}{2} & 1 \\ 0 & 1 & 0 & \frac{1}{2} & -1 & -\frac{1}{2} \\ 0 & 0 & 1 & 0 & \frac{1}{2} & 0 \end{bmatrix} \tag{13}$$

$C2 := eval(A1, a = 2)$

$$C2 := \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 0 & -4 & 0 & -2 & 1 & 0 \\ 0 & 0 & -4 & -1 & 0 & 1 \end{bmatrix} \tag{14}$$

$$C3 := \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{4} \\ 0 & -\frac{1}{4} & 0 \\ 0 & 0 & -\frac{1}{4} \end{bmatrix} C2$$

$$C3 := \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 1 & \frac{1}{4} & 0 & -\frac{1}{4} \end{bmatrix} \tag{15}$$

Hence the inverse for $a = 0$ is

$$\frac{1}{4}\begin{bmatrix} 0 & 6 & 4 \\ 2 & -4 & -2 \\ 0 & 2 & 0 \end{bmatrix}$$

and the inverse for $a = 2$ is

$$\frac{1}{16}\begin{bmatrix} -4 & 8 & 4 \\ 8 & -4 & 0 \\ 4 & 0 & -4 \end{bmatrix}$$

both of which agree with the formula for general $a \neq -2$.

Grading

- A correct setup with significant steps shown is worth 9 **bonus points** in the

Daily Grades category,

- A nearly-correct answer (but for a small error) is worth **12 bonus points** ", and
- A correct answer (with correct work, of course) is worth **14 bonus points**.

Question 2c [extra credit]

An answer that shows knowledge of how to correctly multiply matrices is worth **2 bonus points**

Question 2d [extra credit]

$$C := eval\left(\frac{1}{(a+2)^2}\begin{bmatrix} -2\,a & a+6 & 4 \\ 3\,a+2 & -4 & -2+a \\ a^2 & -a+2 & -2\,a \end{bmatrix}, a=-1\right)$$

$$C := \begin{bmatrix} 2 & 5 & 4 \\ -1 & -4 & -3 \\ 1 & 3 & 2 \end{bmatrix} \tag{16}$$

Since the original matrix equation is

$$\mathbf{A}\boldsymbol{x} = \mathbf{b}$$

the solution is obtained by multiplying both sides on the left by $\mathbf{A}^{-1}$ to obtain

$$\boldsymbol{x} = \mathbf{A}^{-1}\mathbf{b} = (1)\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 5 \\ -4 \\ 3 \end{bmatrix} + (1)\begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

where we performed the matrix multiplication by taking a linear combination of the columns of $\mathbf{A}^{-1}$.

Correct setup and use of the matrix inverse and reduced row echelon to solve the equation each earn 3 bonus points, even if your answer to 3b was incorrect.

All bonus points will be applied to the Daily Grades category.

Total bonus points for this quiz are capped at 20.