Question 1

We must solve the quadratic equation

$$\gamma^2 - 4(3\text{ kg})(3\text{ N/m}) = 0 \quad (1)$$
$$\gamma^2 = 4(3\text{ kg})(3\text{ kg} \cdot s^{-2}) \quad (2)$$

to obtain $\boxed{\gamma = 6\text{ kg / s}}$.

The expression of the function takes the form

$$u(t) = (At + B)e^{-\gamma/(2m)t} = u(t) = (At + B)e^{-t}$$

and solving for the coefficients A and B is a matter of solving a system:

$$\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} = \begin{bmatrix} B \\ A - B \end{bmatrix} = \begin{bmatrix} .25 \\ 0 \end{bmatrix}$$

Hence

$$u(t) = \frac{1}{4}(t+1)e^{-t}$$

The exponential e^{-t} never vanishes and $t + 1$ never vanishes for positive t, hence $u(t)$ has no t-intercepts for $t > 0$.

The simplified version of $u''(t)$ is given by

$$u''(t) = \frac{1}{4}(t-1)e^{-t}$$

Two of the factors in this product are always positive and one (*i.e.*, $t - 1$) changes from negative to positive at $t = 1$. So $t = 1$ is a point of inflection where $u(t)$ changes from concave down to concave up, as Figure 1 shows.

Question 2

As usual, we solve the homogeneous equation first

$$u_h := A\cos(9t) + B\sin(9t) \quad (3)$$

then solve the non-homogeneous. We use complexification, setting

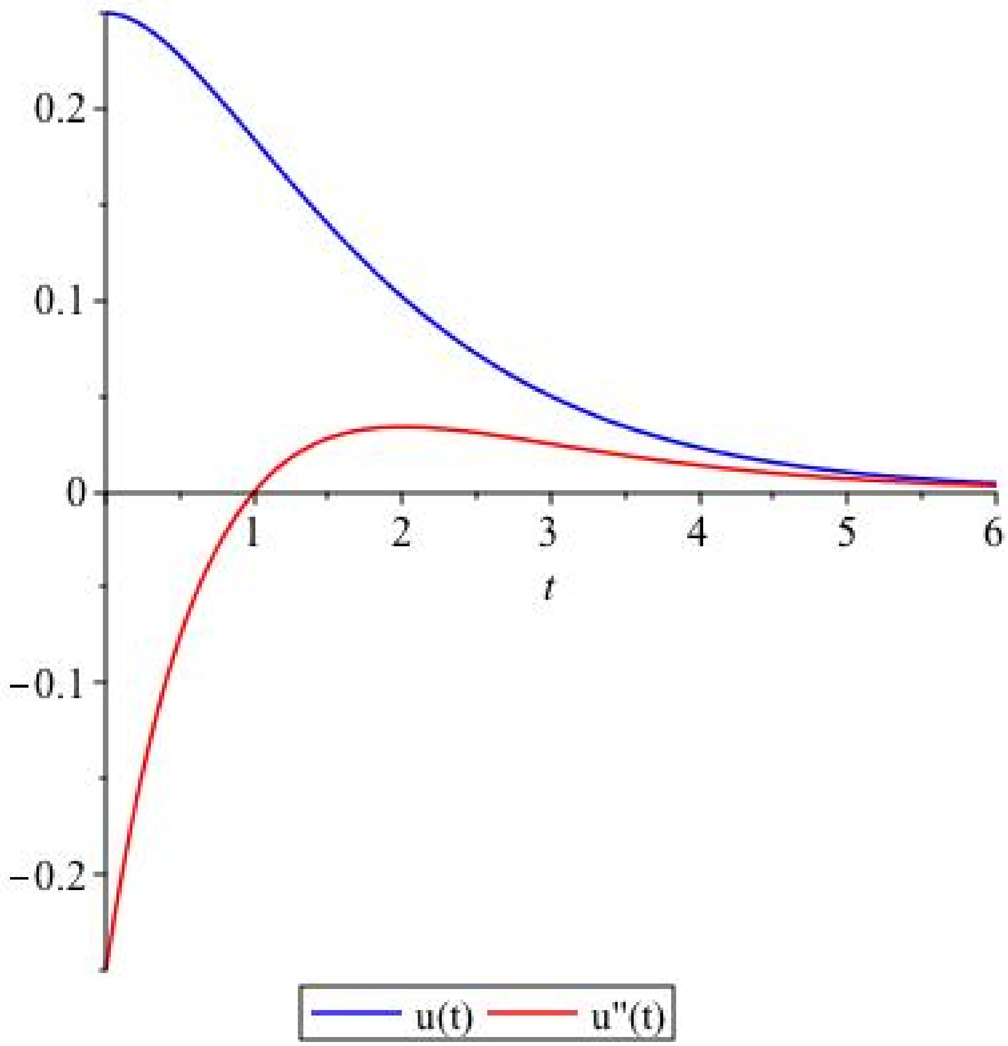
$$y_p := Ce^{i11t} \quad (4)$$

and obtaining through derivation

$$-40C * e^{i11t} = 40e^(i11t) \quad (5)$$

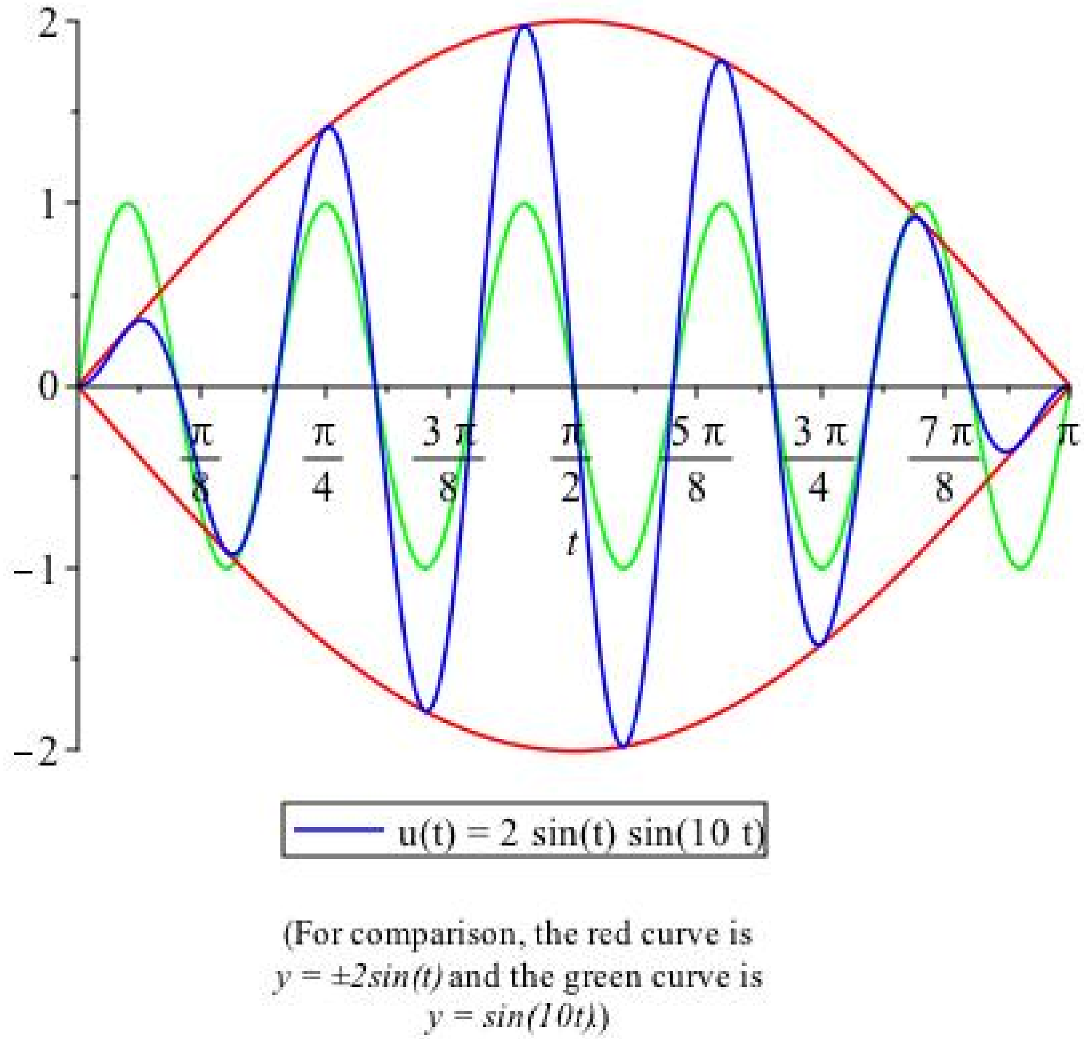
whence

$$C = -1 \quad (6)$$
$$y_p = -e^{i11t} \quad (7)$$
$$u_p = \Re(y_p) = -\cos(11t) \quad (8)$$



t

Figure 1: Graph of the spring system in Question 1.



t

Figure 2: Graph of $u(t)$ from Question 2.

So

$$u(t) = A\cos(9t) + B\sin(9t) - \cos(11t) \tag{9}$$

We solve the system

$$\begin{bmatrix} u(0) \\ u'(0) \end{bmatrix} = \begin{bmatrix} A-1 \\ 9B \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

And so

$$\begin{aligned} u(t) &= \cos(9t) - \cos(11t) && (10)\\ &= \cos(10t - t) - \cos(10t + t) && (11)\\ &= \big[\cos(10t)\cos(t) + \sin(10t)\sin(t)\big] - \big[\cos(10t)\cos(t) - \sin(10t)\sin(t)\big] && (12)\\ &= 2\sin(10t)\sin(t) && (13) \end{aligned}$$

Our function should satisfy the following requirements:

- have amplitude less than or equal to two
- have amplitude bounded above and below by the sine curve $2sint$ (because $|sin(10t)| < 1$
- have a "frequency" that resembles that of $\sin(10t)$

Indeed, we see these expectations borne out in the actual graph, shown in Figure 2.

Question 3 (Since the method is essentially the same for all six, only the method for the first listed will be copied here. We also go about the process a bit more laboriously than necessarily for the purpose of instruction; you will not have needed to be as detailed to get full credit.)

We start with the matrix

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & 2 \\ -1 & 3 & 1 \end{bmatrix}$$

We will choose the second row as our "pivot row" since it is simple and contains the most zeroes.

We list each element in this row *in order* along with its corresponding *minor* (*i.e.*, the matrix that is formed by crossing out the row and column that contains the element we're working with).

First, at row 2 column 1 (total 3), is element **0** with minor

$$\begin{bmatrix} - & 2 & 1 \\ - & - & - \\ - & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}$$

which has determinant $(2)(1) - (3)(1) = -1$. The product of the element and the determinant of its minor is thus $0 \cdot -1 boxed{=} 0$.

In the next column is element **2** with minor

$$\begin{bmatrix} 1 & - & 1 \\ - & - & - \\ -1 & - & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

which has determinant $(1)(1) - (-1)(1) = 2$. The product $2 \cdot 2 = 4$.

In the final column is element **2** with minor

$$\begin{bmatrix} 1 & 2 & - \\ - & - & - \\ -1 & 3 & - \end{bmatrix} \to \begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}$$

which has determinant $(1)(3) - (2)(-1) = 5$. The product $2 \cdot 5 = 10$.

To find the determinant of the matrix, we take the sum of the product on this list subject to the following caveats:

- We switch the sign of every other summand, and
- At the end of the sum, we multiply by $(-1)^k$, where k is the total of the coordinate of the row and column that we started at (3 in this case)

And so the determinant of our matrix is

$$(-1)^3\big[(1)(0)+(-1)(4)+(1)(10)\big] \tag{14}$$
$$-\big[(0)+(-4)+(10)\big] \tag{15}$$
$$=-(-6) \tag{16}$$
$$\boxed{-6} \tag{17}$$

The other determinants are, in order:

- -18 (multiplying a row or column by a scalar k multiplies the determinant by the same scalar)
- 9 (see above)
- 6 (interchanging two rows or columns multiplies the determinant by -1)
- -6 (see above)
- -6 (adding a multiple of one row to another row has no effect on determinant)

You may cite the above rules on an exam or quiz to help calculate determinants.