Quiz 5

Find the particular solution to the initial value problem

$$y'' + 2y' + 5y = e^{-t}; \quad y(\pi/4) = \frac{3}{4}e^{-\pi/4},\ y'(\pi/4) = 0$$

Solution The characteristic polynomial is

$$\lambda^2 + 2\lambda + 5 = 0$$

which has roots

$$\lambda = \frac{1}{2}\big(-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 5}\big) \tag{1}$$
$$= \frac{1}{2}\big(-2 \pm 4i\big) \tag{2}$$
$$= -1 \pm 2i \tag{3}$$

Hence

$$y_h = e^{-t}\big(A\cos(2t) + B\sin(2t)\big) \tag{4}$$

We use the method of undetermined coefficients and guess

$$y_p = ce^{-t} \tag{5}$$

to obtain

$$y_p'' + 2y_p' + 5y_p = (c - 2c + 5c)e^{-t} \tag{6}$$
$$= 4ce^{-t} \tag{7}$$

Comparing to the RHS, we obtain

$$4ce^{-t} = e^{-t} \tag{8}$$
$$c = \frac{1}{4} \tag{9}$$

whence

$$y_p = \frac{1}{4}e^{-t} \tag{10}$$

$$y(t) = y_h + y_p = e^{-t}\big(A\cos(2t) + B\sin(2t) + \frac{1}{4}\big) \tag{11}$$

$$y'(t) = [(2B - A)\cos(2t) - \frac{1}{4} + (-2A - B)\sin(2t)]e^{-t} \tag{12}$$

$$y(\pi/4) = e^{-\pi/4}(B + \frac{1}{4}) = \frac{3}{4}e^{-\pi/4} \tag{13}$$

$$y'(\pi/4) = -\frac{1}{4}e^{-\pi/4}(4B + 8A + 1) = 0 \tag{14}$$

From lines 13 and 14 we obtain the system of equations

$$4B = 2 \tag{15}$$

$$8A + 4B = -1 \tag{16}$$

$$\tag{17}$$

So the solution is

$$\boxed{y(t) = e^{-t}\Big(- \frac{3}{8}\cos(2t) + \frac{1}{2}\sin(2t) + \frac{1}{4}\Big)}$$