Quiz 2 Rubric

Question 1 [6pts] Consider the differential equation

$$ty' + (\cos t)y = |t-1|$$

with initial conditions (ordered pairs are of the form (t, y)):

Determine how many solutions Theorem 2.4.1 guarantees for each initial condition. If the theorem guarantees a unique solution, indicate the largest interval on which the theorem guarantees that solution to satisfy the DE and be unique.

Answer First of all, we must get the DE into canonical form:

$$y' + \frac{\cos t}{t} y = \frac{|t-1|}{t}$$

We see that [1pt]

$$p(t) = \frac{\cos t}{t}$$
$$q(t) = \frac{|t-1|}{t}$$

These functions fail to be continuous only at $t = 0$, so our two choices of interval are $(-\infty, 0)$ and $(0, \infty)$.

(a) (0,1)

We have $t_0 = 0$, a value that is not contained in either interval over which both $p(t)$ and $q(t)$ are continuous. So 2.4.1 fails to apply and hence says nothing about the number of solutions with this IC [2pt].

(b) (1,0)

We have $t_0 = 1$, so $t_0 \in (0, \infty)$. So 2.4.1 says that there exists a unique function $y = \phi(t)$ that satisfies the DE on $(0, \infty)$ and passes through $(1, 0)$; i.e., 2.4.1 guarantees **exactly one** solution [2pt] for $t > 0$ [1pt].

Question 2 [14 pts] For each of the following differential equations with initial conditions, apply Theorem 2.4.2 to make a statement about existence and uniqueness of solutions passing through the given point; then give the *exact* number of solutions that do so.

(a) $\frac{dy}{dt} = \frac{1}{y}, \quad (1, 0)$

Any rectangle around $(1, 0)$ must include points where $y = 0$, so $f(t, y) := 1/y$ is not continuous in any rectangle around $(1, 0)$ and **2.4.2 makes no guarantees** [3pts]. We can tell that there are **no solutions** [1pts] to this DE because we cannot have a derivative of $1/0$ at $(1, 0)$.

(b) $\frac{dy}{dt} = \sqrt{y}, \quad (0, 0)$

Setting $f(t, y) := \sqrt{y}$, we have that $\frac{\delta f}{\delta y} = \frac{1}{2\sqrt{y}}$ [2pts]. Hence, the theorem makes a statement about *existence* but none about *uniqueness* [3pts]. Two solutions [e.c.] through this point are

$$y_1(t) \equiv 0$$
$$y_2(t) = \begin{cases} 0, & t < 0 \\ t^2/4, & t \geq 0 \end{cases}$$

(c) $\frac{dy}{dt} = \sqrt{y}, \quad (0, 9)$

We can draw a rectangle $[\alpha, \beta] \times [\gamma, \delta]$ around $(0, 9)$ in which f and f_y are continuous (all we have to do is avoid the line $y = 0$). So, on some interval $\alpha < t_0 - h < t < t_0 + h < \beta$ [2pts], we have that a **unique solution** [3pt] to the DE with initial condition exists.

EDIT: No one got the points about the interval $(t_0 - h, t_0 + h)$, so I decided to give +2 points back to all the quizzes. However, you should be aware that when 2.4.2 guarantees existence/uniqueness **it does so only "close" to the IC**. In other words, the solution need not be unique in the entire rectangle around (t_0, y_0).

Other comments I expected explicit applications of theorems (*e.g.*, "2.4.1 applies *because* ...") and so took off at least one point for mere yes/no answers to the theorem application questions.