HW 7.4

6 (a) $\begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix} = (2t)(t) - (t^2)(1) = 2t^2 - t^2 = \boxed{t^2}$

(b) The columns are LI except at $t=0$ (i.e., where the Wronskian is zero)

Hence the columns are LI on every interval.

(c)

$$x' = Ax$$

$$\begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix} = A \begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}$$

$$\begin{pmatrix} t & t^2 \\ 1 & 2t \end{pmatrix}^{-1} = \frac{1}{t \cdot 2t - t^2 \cdot 1} \begin{pmatrix} 2t & -t^2 \\ -1 & t \end{pmatrix} = \frac{1}{t^2} \begin{pmatrix} 2t & -t^2 \\ -1 & t \end{pmatrix}$$

$$A = \begin{pmatrix} 1 & 2t \\ 0 & 2 \end{pmatrix} \begin{pmatrix} 2t & -t^2 \\ -1 & t \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 1 \\ -\frac{2}{t^2} & \frac{2}{t} \end{pmatrix}$$

HW 7.5

3 $\begin{vmatrix} 2-\lambda & -1 \\ 3 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda) - (-1)(3)$

$$= (\lambda^2 - 4) - (-3)$$

$$= \lambda^2 - 1$$

Eigenvalues $\boxed{\lambda = \pm 1}$

(a) $\lambda = 1$ Eigenvector

We solve

$$(A - I)v = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix} v = 0$$

$$\boxed{v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}} \checkmark$$

$\lambda = -1$

$$(A + I)v = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix} v = 0$$

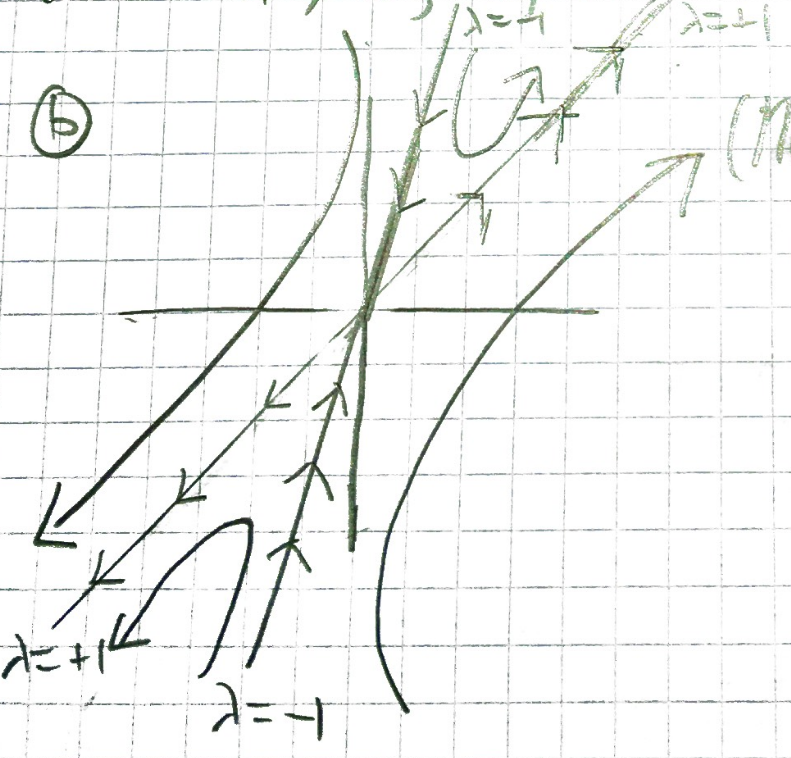
$$\boxed{v = \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

So the general solution is

$$\boxed{y(t) = c_1 e^{t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

When $c_1 \neq 0$, $|y(t)| \to \infty$ as $t \to \infty$.

(b)



(No arrows for axes unless they are straight line solutions)

7.5

(12) $\begin{vmatrix} 3-\lambda & 2 & 4 \\ 2 & -\lambda & 2 \\ 4 & 2 & 3-\lambda \end{vmatrix} = \lambda^3 - 6\lambda^2 - 15\lambda - 8$

$= (\lambda-8)(\lambda+1)^2$

[NOTE that the root -1 can be found through adding coefficients with alternating signs:

$$\begin{array}{cccc} \lambda^3 & -6\lambda^2 & -15\lambda & -8 \\ \downarrow & \downarrow & \downarrow & \downarrow \\ 1 & -6 & -15 & -8 \\ + & - & + & - \\ (1) + & (6) + & (-15) + & (8) = 0 \checkmark \end{array}$$
]

$\lambda = -1$:

$$(A+I)v = \begin{pmatrix} 4 & 2 & 4 \\ 2 & 1 & 2 \\ 4 & 2 & 4 \end{pmatrix} v = 0$$

Two LI solutions: $v_1 = \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix}$

$\lambda = 8$:

$$(A-8I) = \begin{pmatrix} -5 & 2 & 4 \\ 2 & -8 & 2 \\ 4 & 2 & -5 \end{pmatrix} v = 0$$

see note on next page.

$$v = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$$

So the G.S is

$$x(t) = \left[\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix} c_1 + \begin{pmatrix} 0 \\ -2 \\ 1 \end{pmatrix} c_2 \right] e^{-t} + \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} c_3 e^{8t}$$

7.5 cont

12 cont (NOTE: Any two LI vectors in the eigenspace of $\lambda = -1$ ~~(then~~ (set of vectors that are linear combinations of two fixed eigenvectors) of $\lambda=-1$ will do here as eigenvectors of $\lambda = -1$.)

16
$$\begin{vmatrix} -2-\lambda & 1 \\ -5 & 4-\lambda \end{vmatrix} = (-2-\lambda)(4-\lambda)+5$$
$$= (\lambda-4)(\lambda+2)+5$$
$$= \lambda^2-2\lambda-8+5$$
$$= \lambda^2-2\lambda-3$$
$$=(\lambda-3)(\lambda+1)$$

$\lambda=-1$

$\begin{pmatrix} -1 & 1 \\ -5 & 5 \end{pmatrix} v = 0 \to v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 3$

$\begin{pmatrix} -5 & 1 \\ -5 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 5 \end{pmatrix} = 0 \to v = \begin{pmatrix} 1 \\ 5 \end{pmatrix}$

$$x(t) = c_1 e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 e^{3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}$$

$$x(0) = c_1\begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2\begin{pmatrix} 1 \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\boxed{= \frac{1}{2}\left(e^{-t}\begin{pmatrix} 1 \\ 1 \end{pmatrix} + e^{3t}\begin{pmatrix} 1 \\ 5 \end{pmatrix}\right)}$$

$$\left[\begin{array}{cc|c} 1 & 1 & 1 \\ 1 & 5 & 3 \end{array}\right]$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{4}\begin{pmatrix} 5 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 5+3(-1) \\ -1+3(1) \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 2 \\ 2 \end{pmatrix}$$
$$= \begin{pmatrix} \frac{1}{2} \\ \end{pmatrix}$$

4.3 P1

y_H: The CP is

$$\lambda^3+\lambda^2+\lambda+1$$

$$=(\lambda^2+1)(\lambda+1) \qquad \lambda=\pm i,\ -1$$

so $y_H(t) = ae^{-t} + b\cos(t) + c\sin(t)$.

y_p: Guess

$$Ate^{-t} + Bt + C$$

$$y_p = \cancel{Ate^{-t}} + Bt + C$$

$$y_p' = \cancel{-Ate^{-t} + Ae^{-t}} + B$$

$$y_p'' = \cancel{Ate^{-t} - 2Ae^{-t}}$$

$$y_p''' = \cancel{-Ate^{-t} \quad 3Ae^{-t}} \qquad A=\tfrac{1}{2}$$

"$2Ae^{-t} = e^{-t}$"

$$y_p''' + y_p'' + y_p' + y_p = Bt + (B+C)$$

$$= 4t + 0$$

$$B=4 \qquad C=-4$$

$$\boxed{y = y_H + y_p = (ae^{-t} + b\cos t + c\sin t) + \left(\tfrac{1}{2}\cdot te^{-t} + 4t - 4\right)}$$

(9) $\underline{\underline{CP}}$ $\lambda^3+4\lambda = \lambda(\lambda^2+4)=0$

$y_H(t) = a\cdot e^{0t} + b\cdot\cos(2t) + c\sin(2t)$

Initial guess
$y_p(t) := At+B$

$\underline{\underline{BUT}}$ $y=-\frac{t}{4}+k$ solves the homogeneous equation
Hence we must multiply by t^2 to obtain a
guess of $y_p := At^3+Bt^2$

$$\begin{aligned} y_p &= At^3 + Bt^2 \\ y_p' &= 3At^2 + 2Bt \\ y_p'' &= 6At + 2B \\ y_p''' &= 6A \end{aligned}$$

$$t = y_p''' + 4y_p' = 6A + 4(3At^2+2Bt)$$
$$= 6A + 12At^2 + 8Bt$$

$0=6A$
$0=12A$
$1=8B$
$B=\frac{1}{8}$ $A=0$ $y_p = \frac{t^2}{8}$

$\boxed{y \overset{G.S}{=} a+b\cos(2t)+c\sin(2t)+\frac{t^2}{8}}$

P8

$y = a + b\cos(2t) + c\sin(2t) + \frac{t^2}{8}$

$y(0) = a + b = 0$

$y'(t) = -2b\sin(2t) + 2c\cos(2t) + \frac{t}{4}$

$y'(0) = 2c = 0 \rightarrow c = 0$

$y''(t) = -4b\cos(2t) + \frac{1}{4}$

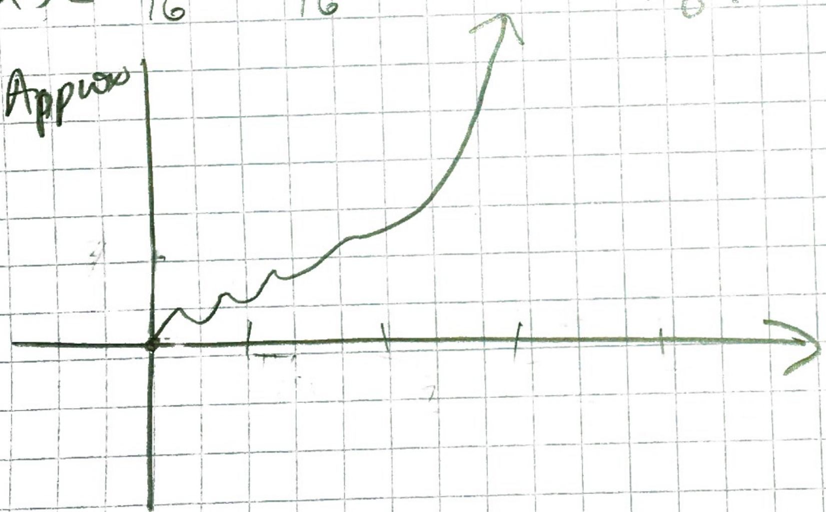
$y''(0) = -4b + \frac{1}{4} = 1$

$-4b = \frac{3}{4}$

$b = -\frac{3}{16}$

$a = \frac{3}{16}$

$y(t) = \frac{3}{16} - \frac{3}{16}\cos(2t) + \frac{t^2}{8}$



⑫ 1 2 1 8 −12 sum to 1

so 1 is a root of the polynomial.

By synthetic division:

	1	2	1	8	−12
		1	3	4	12
1	1	3	4	12	0

~~Poly~~ $C.P = (x-1)(x^3+3x^2+4x+12)$
$= (x-1)(x+3)(x^2+4)$

$$y_H(t) = ae^{t} + be^{-3t} + c\cdot\cos(2t) + d\cdot\sin(2t)$$

$y_{P1} := Ae^{-t} \longrightarrow -20Ae^{-t} = -e^{-t} \Rightarrow A = \frac{1}{20}$

$y_{P2} := Be^{it} \longrightarrow (-12+6i)B\cancel{e^{it}} = 12\cancel{e^{it}}$

$$B = \frac{12}{-12+6i}$$

since the RHS contains $\sin t$, we take imaginary part of y_{P2} to obtain

$$y_p = \frac{1}{4}e^{-t} + \left(-\frac{2}{5}\cos t - \frac{4}{5}\sin t\right)$$

$y(t) =$ (use a computer for the remainder)