Chapter 3

3.1

3.1.10

The characteristic polynomial is

$$\lambda^2 + 4\lambda + 3 = (\lambda + 3)(\lambda + 1)$$

and so

$$\begin{align} y(t) &= Ae^{-t} + Be^{-3t} \tag{3.1} \\ y'(t) &= -Ae^{-t} - 3Be^{-3t} \tag{3.2} \\ 2 &= A + B \tag{3.3} \\ -1 &= -A - 3B \tag{3.4} \end{align}$$

This system gives $A = \frac{5}{2}$ and $B = -\frac{1}{2}$, hence

$$y(t) = \frac{1}{2}(5e^{-t} - e^{-3t})$$

As $t \to \infty$, $y(t) = e^{-t}\frac{1}{2}(5 - e^{-2t}) \to 0 \cdot \frac{5}{2} = 0$.

As $t \to -\infty$, $y(t) = e^{-3t}\frac{1}{2}(5e^{2t} - 1) \to \infty \cdot (-1) = -\infty$.

Setting $\frac{1}{2}(5e^{-t} - e^{-3t}) = 0$, we find a t-intercept with $t < 0$.

For large t, $y'(t) = -e^{-t}\frac{1}{2}(5 - 3e^{-2t})$ is negative. Hence we obtain the following graph:

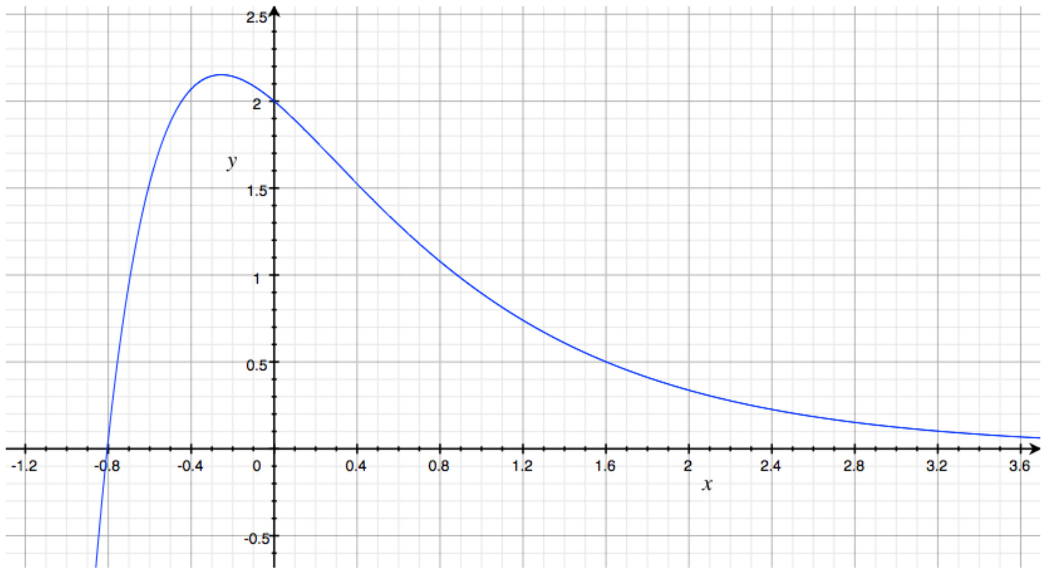

Figure 3.1: The solution to Problem 10 of Chapter 3.1.

3.1.18

The differential equation

$$2y'' + 5y' + 2y = 0$$

has the characteristic polynomial

$$2\lambda^2 + 5\lambda + 2\lambda = 0$$

which has roots $-1/2, 2$.

3.1.21

The characteristic polynomial is

$$\lambda^2 - \lambda - 2$$

, which has roots $\lambda = -1, 2$. Hence the general solution is

$$y(t) = Ae^{-t} + Be^{2t}$$

with derivative

$$y'(t) = -Ae^{-t} + 2Be^{2t}$$

We must now solve the system of equations

$$\alpha = A + B \tag{3.5}$$

$$2 = -A + 2B \tag{3.6}$$

We obtain $B = \frac{1}{3}(\alpha + 2)$ and $A = \frac{2}{3}(\alpha - 1)$ through a simple addition of the equations. If $B \neq 0$ then $|y(t)| \to \infty$ as $t \to \infty$; on the other hand $\beta = 0$ results in a limit of zero.

3.2

3.2.2

$$\begin{vmatrix} \cos(t) & -\sin(t) \\ \sin(t) & \cos(t) \end{vmatrix} = \cos^2 t + \sin^2 t = \boxed{1}$$

3.2.7

$$p(t) = 0 \tag{3.7}$$

$$q(t) = 3/t \tag{3.8}$$

$$g(t) = 1 \tag{3.9}$$

So the largest interval where a unique solution is guaranteed is $(0, \infty)$ [we have continuity on $(-\infty, 0)$ also but $1 \notin (-\infty, 0)$ so we are not interested in that interval].

3.2.24

We have

$$
\begin{aligned}
y_1(t) &= \cos(2t) && (3.10)\\
y_1''(t) &= -4\cos(2t) && (3.11)\\
y_2(t) &= \sin(2t) && (3.12)\\
y_2''(t) &= -4\sin(2t) && (3.13)
\end{aligned}
$$

so it is clear that the DEs are satisfied. The Wronskian is

$$
\begin{vmatrix} \cos(2t) & -2\sin(2t) \\ \sin(2t) & 2\cos(2t) \end{vmatrix} = 2\cos^2(2t) + 2\sin^2(2t) = \boxed{2}
$$

So the two solutions make a fundamental set.

3.2.29

Using the notation of the book,

$$
p(t) = -(1 + 2/t) \qquad (3.14)
$$

So we calculate

$$
\begin{aligned}
\int p(t)\, d(t) &= -\int 1 + 2/t\, dt && (3.15)\\
&= -(t + 2\ln|t| + c) && (3.16)\\
W(t) &= Ce^{-\int p(s)\, ds} && (3.17)\\
&= \boxed{Ct^2 e^t} && (3.18)
\end{aligned}
$$

3.2.33

Simply apply the product rule to the first term then divide both sides by $p(t)$.

3.3

3.3.4

$$e^{2-\frac{\pi}{2}i} = e^2\Big[\cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})\Big] = e^2[0-i] = \boxed{-e^2}$$

3.3.16

Characteristic polynomial:

$$\lambda^2 + 4\lambda + 6.25 = (\lambda - (-2+1.5i))(\lambda - (-2-1.5i))$$

Thus the general solution is

$$\boxed{y(t) = e^{-2t}\Big[A\cos(\frac{3}{2}t) + B\sin(\frac{3}{2}t)\Big]}$$

3.3.22

Characteristic polynomial:

$$\lambda^2 + 2\lambda + 2 = (\lambda - (-1+i))(\lambda - (-1-i))$$

Thus the general solution is

$$y(t) = e^{-t}\Big[A\cos(t) + B\sin(t)\Big] \tag{3.19}$$

which gives

$$y'(t) = e^{-t}\Big[-A\sin(t) + B\cos(t)\Big] - e^{-t}\Big[A\cos(t) + B\sin(t)\Big] \tag{3.20}$$

The initial conditions give

$$2 = y(\pi/4) = e^{-\pi/4}\Big[A\cos(\pi/4) + B\sin(\pi/4)\Big] \tag{3.21}$$

$$= \frac{\sqrt{2}}{2}e^{-\pi/4}\Big[A + B\Big] \tag{3.22}$$

$$2\sqrt{2}e^{\pi/4} = A + B \tag{3.23}$$

$$-2 = y'(\pi/4) = e^{-\pi/4}\Big[-A\sin(\pi/4) + B\cos(\pi/4)\Big] - e^{-\pi/4}\Big[A\cos(\pi/4) + B\sin(\pi/4)\Big] \tag{3.24}$$

$$= e^{-\pi/4}\frac{\sqrt{2}}{2}\Big[-A + B\Big] - e^{-\pi/4}\frac{\sqrt{2}}{2}\Big[A + B\Big] \tag{3.25}$$

$$2\sqrt{2}e^{\pi/4} = 2A \tag{3.26}$$

$$\sqrt{2}e^{\pi/4} = A = B \tag{3.27}$$

$$y(t) = \sqrt{2}e^{(\pi/4)-t}\Big(\cos(t) + \sin(t)\Big) \tag{3.28}$$

It oscillates with decaying magnitude.

3.3.24

The roots of the characteristic polynomial are $\lambda = \frac{1}{5}(-1 \pm i\sqrt{34})$, so the general solution is

$$u(t) = e^{-t/5}\Big(A\cos(\frac{\sqrt{34}}{5}t) + B\sin(\frac{\sqrt{34}}{5}t)\Big) \tag{3.29}$$

Taking the derivative,

$$u'(t) = \frac{\sqrt{34}}{5}e^{-t/5}\Big(-A\sin(\frac{\sqrt{34}}{5}t) + B\cos(\frac{\sqrt{34}}{5}t)\Big) - \frac{1}{5}e^{-t/5}\Big(A\cos(\frac{\sqrt{34}}{5}t) + B\sin(\frac{\sqrt{34}}{5}t)\Big) \tag{3.30}$$

We apply the initial conditions to obtain

$$2 = u(0) = A \tag{3.31}$$

$$1 = u'(0) = \frac{\sqrt{34}}{5}B - \frac{2}{5} \tag{3.32}$$

Hence

$$u(t) = e^{-t/5}\Big(2\cos(\frac{\sqrt{34}}{5}t) + \frac{7}{\sqrt{34}}\sin(\frac{\sqrt{34}}{5}t)\Big) \tag{3.33}$$

3.4

3.4.2

The characteristic polynomial is $9\lambda^2 + 6\lambda + 1$, which factors as

$$9\lambda^2 + 6\lambda + 1 = (3\lambda + 1)^2 = 0$$

so as to yield root $\lambda = -1/3$ with multiplicity 2. Hence the general solution is

$$y(t) = (At + B)e^{-t/3}$$

3.4.14

The general solution is

$$y(t) = (At + B)e^{-2t} \tag{3.34}$$

with derivative

$$\begin{aligned} y'(t) &= (At + B)(-2)e^{-2t} + Ae^{-2t} && (3.35)\\ &= (-2At - 2B + A)e^{-2t} && (3.36)\\ 2 = y(-1) &= (-A + B)e^{2} && (3.37)\\ 1 = y'(-1) &= (3A - 2B)e^{2} && (3.38) \end{aligned}$$

whence

$$\begin{aligned} A &= 5e^{-2} && (3.39)\\ B &= 7e^{-2} && (3.40) \end{aligned}$$

So the solution is

$$y(t) = (5t + 7)e^{-2t-2} \tag{3.41}$$

It's easy to see that $y \to 0$ as $t \to \infty$; a denominator with exponential e^{2t+2} will grow much faster than a numerator that's a polynomial of degree 1.

3.4.23

$$
\begin{aligned}
(t^2v)' &= 2tv + t^2v' && (3.42)\\
(t^2v)'' &= 2tv' + 2v + 2tv' + t^2v'' && (3.43)\\
&= t^2v'' + 4tv' + 2v && (3.44)\\
0 = t^2y'' - 4ty' + 6y &= t^2(t^2v'' + 4tv' + 2v) - 4t(2tv + t^2v') + 6(t^2v) && (3.45)\\
&= t^4v'' + 4t^3v' + 2t^2v - 8t^2v - 4t^3v' + 6t^2v && (3.46)\\
0 &= t^4v'' && (3.47)
\end{aligned}
$$

Since $t > 0$,

$$0 = v'' \tag{3.48}$$

Hence

$$
\begin{aligned}
v(t) &= at + b && (3.49)\\
y(t) &= at^3 + bt^2 && (3.50)
\end{aligned}
$$

A new solution that would form a fundamental solution set with t^2 would be t^3.

3.5

3.5.4

The characteristic polynomial is $\lambda^2 + \lambda - 6 = (\lambda + 3)(\lambda - 2) = 0$, hence

$$y_h = Ae^{2t} + Be^{-3t} \tag{3.51}$$

First we tackle the non-homogeneous equation with $g(t) = 12e^{3t}$:

$$
\begin{aligned}
y_3 &:= c_1e^{3t} && (3.52)\\
y_3'' + y_3' - 6y_3 &= 9c_1e^{3t} + 3c_1e^{3t} - 6c_1e^{3t} && (3.53)\\
&= 6c_1e^{3t} && (3.54)\\
&= 12e^{3t} && (3.55)\\
\therefore c_1 &= 2 && (3.56)
\end{aligned}
$$

Then the other:

$$
\begin{aligned}
y_{-2} &:= c_2 e^{-2t} && (3.57)\\
y''_{-2} + y'_{-2} - 6y_{-2} &= 4c_2 e^{-2t} - 2c_2 e^{-2t} - 6c_2 e^{-2t} && (3.58)\\
&= -4c_2 e^{3t} = 12e^{3t} && (3.59)\\
\therefore c_2 &= -3 && (3.60)
\end{aligned}
$$

Combining, we obtain

$$
y_p = 2e^{3t} - 3e^{-2t} \boxed{y(t) = y_h + y_p = Ae^{2t} + Be^{-3t} + 2e^{3t} - 3e^{-2t}} \tag{3.61}
$$

3.5.10

$$
y_h = A\sin t + B\cos t \tag{3.62}
$$

Let y_1 signify the particular solution corresponding to the sine term, y_2, to the other:

$$
\begin{aligned}
y_1 &:= c_1 \sin(2t) + k_1 \cos(2t) && (3.63)\\
y_1'' &:= -4\big(c_1 \sin(2t) + k_1 \cos(2t)\big) && (3.64)\\
y_1'' + y_1 &= (-3c_1)\sin(2t) + (-3k_1)\cos(2t) && (3.65)\\
&= 3\sin(2t) && (3.66)\\
\therefore (c_1, c_2) &= (-1, 0) && (3.67)\\
y_1(t) &= -\sin(2t) && (3.68)
\end{aligned}
$$

We solve for y_2 via complexification. By Euler's formula:

$$
t\cos(2t) = Re\big(te^{i2t}\big) \tag{3.69}
$$

And so we continue like so:

$$y_2(t) := (ct+k)e^{i2t} \tag{3.70}$$
$$y_2''(t) + y_2(t) = (-3ct - 3k + 4ic)e^{i2t} \tag{3.71}$$
$$= te^{i2t} \tag{3.72}$$
$$\therefore -3c = 1 \tag{3.73}$$
$$-3k - \frac{4i}{3} = 0 \tag{3.74}$$
$$k = -\frac{4i}{9} y_2(t) \qquad := (-\frac{1}{3}t - \frac{4}{9}i)e^{i2t} \tag{3.75}$$
$$= (-\frac{1}{3}t - \frac{4}{9}i)(\cos(2t) + i\sin(2t)) \tag{3.76}$$

Taking real parts,

$$\Re y_2 = -\frac{1}{3}t\cos(2t) + \frac{4}{9}\sin(2t) \tag{3.77}$$

So the general solution is

$$y(t) := A\sin t + B\cos t - \sin(2t) - \frac{1}{3}t\cos(2t) + \frac{4}{9}\sin(2t) \tag{3.78}$$
$$= A\sin t + B\cos t - \frac{5}{9}\sin(2t) - \frac{1}{3}t\cos(2t) \tag{3.79}$$
$$\tag{3.80}$$

3.5.20

$\lambda = \frac{-2\pm\sqrt{4-4\cdot 1\cdot 5}}{2} = -1 \pm 2i$ We solve using complexification.

$$y_p := ce^{(2i-1)t} \tag{3.81}$$
$$y_p'' + 2y_p' + 5y_p = 0 \tag{3.82}$$

So we must multiply by t and try again:

$$y_p := cte^{(2i-1)t} \tag{3.83}$$
$$y_p'' + 2y_p' + 5y_p = 4ice^{(2i-1)t} \tag{3.84}$$
$$c = -i \tag{3.85}$$

So the particular solution is

$$\Re(y_p) = \Re(-ite^{(2i-1)t}) \tag{3.86}$$
$$= te^{-t}\sin(2t) \tag{3.87}$$

Hence the solution, after accounting for initial conditions, is

$$y(t) = e^{-t}\Big(\cos(2t) + \frac{1}{2}\sin(2t) + t\cos(2t)\Big) \tag{3.88}$$

3.6

3.6.1

Based on the characteristic polynomial, $y_1(t) = e^{2t}$ and $y_2(t) = e^{3t}$. You can check that $W(y_1, y_2) = e^{5t}$

$$Y(t) = -e^{2t}\int_{t_0}^{t} \frac{e^{3s}\cdot 2e^{s}}{e^{5s}}\,ds + e^{3t}\int_{t_0}^{t} \frac{e^{2s}\cdot 2e^{s}}{e^{5s}}\,ds \tag{3.89}$$
$$= -e^{2t}\int_{t_0}^{t} 2e^{-s}\,ds + e^{3t}\int_{t_0}^{t} 2e^{-2s}\,ds \tag{3.90}$$
$$= -e^{2t}\cdot(-2e^{-t})\Big|_0^t + e^{3t}\cdot(-e^{-2t})\Big|_0^t \tag{3.91}$$
$$= -e^{2t}\cdot(-2e^{-t}+2) + e^{3t}\cdot(-e^{-2t}+1) \tag{3.92}$$
$$= 2e^{t} - 2e^{2t} - e^{t} + e^{3t} \tag{3.93}$$
$$= \boxed{e^{t} - 2e^{2t} + e^{3t}} \tag{3.94}$$

Trying the method of undetermined coefficients with

$$y_p := ce^{t} \tag{3.95}$$

gives a particular solution of

$$\tag{3.96}$$
$$c = 1 \tag{3.97}$$

which is consistent with our findings (the e^{2t} and e^{3t} terms may be contributed by the homogeneous solution).

3.6.7

The homogeneous part of the general solution is

$$y_h = e^{-2t}(At + B)$$

Call $y_1(t) = te^{-2t}$ and $y_2(t) = e^{-2t}$. Then $W(y_1, y_2) = -2te^{-4t} - (1-2t)e^{-4t} = -e^{-4t}$ a particular solution is given by

$$y_p(t) = -te^{-2t} \int_{t_0}^{t} \frac{e^{-2s}s^{-2}e^{-2s}}{-e^{-4s}}\, ds + e^{-2t} \int_{t_0}^{t} \frac{se^{-2s}s^{-2}e^{-2s}}{-e^{-4s}}\, ds \tag{3.98}$$

We choose a convenient value for t_0:

$$\begin{aligned}
&= -te^{-2t} \int_1^t -\frac{1}{s^2}\, ds + e^{-2t} \int_1^t -\frac{1}{s}\, ds && (3.99)\\
&= -te^{-2t} \cdot -\frac{1}{t}\Big|_1^t + e^{-2t} \cdot - \ln t\Big|_1^t && (3.100)\\
&= (-te^{-2t})(-t^{-1} + 1) - e^{-2t} \ln t && (3.101)\\
&= e^{-2t}(1 - t - \ln t) && (3.102)
\end{aligned}$$

We can throw out the summands that also solve the homogeneous equation to obtain that the particular solution is $Y(t) = -e^{-2t} \ln t$. So the general solution is

$$y_h + y_p = \boxed{e^{-2t}(At + B - \ln t)}$$