Chapter 2

Chapter 2.1

2.1.2

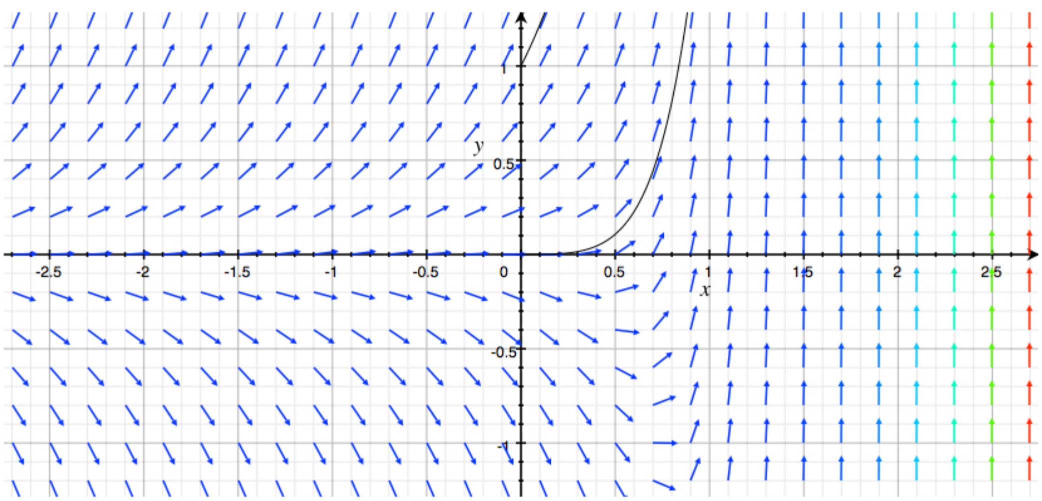

Figure 2.1: Direction field for Chapter 2.1, Problem 2.

$$\mu(t) = e^{-2t} \tag{2.1}$$

$$y(t) = e^{2t} \int e^{-2t} \cdot t^2 e^{2t} dt \tag{2.2}$$

$$= e^{2t}\left[\frac{1}{3}t^3 + C\right] \tag{2.3}$$

As $t \to \infty$, all solutions increase without bound.

2.1.8

Standard form: $y' + \frac{4t}{1+t^2}y = \frac{1}{(1+t^2)^3}$

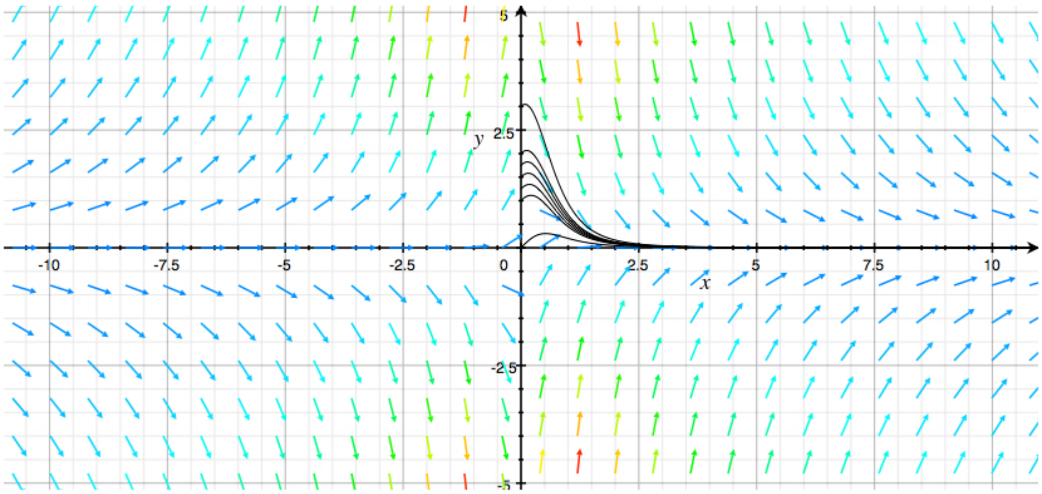

Figure 2.2: Direction field for Chapter 2.1, Problem 8.

$$\mu(t) = e^{\int \frac{4t}{1+t^2} dt} = (1+t^2)^2 \tag{2.4}$$

$$y(t) = \frac{1}{(1+t^2)^2} \int (1+t^2)^2 \frac{1}{(1+t^2)^3} dt \tag{2.5}$$

$$= \frac{1}{(1+t^2)^2} \Big[\arctan t + C \Big] \tag{2.6}$$

$$\tag{2.7}$$

All solutions tend to zero as $t \to \infty$.

2.1.22

Standard form: $y' - \frac{1}{2}y = \frac{1}{2}e^{t/3}$

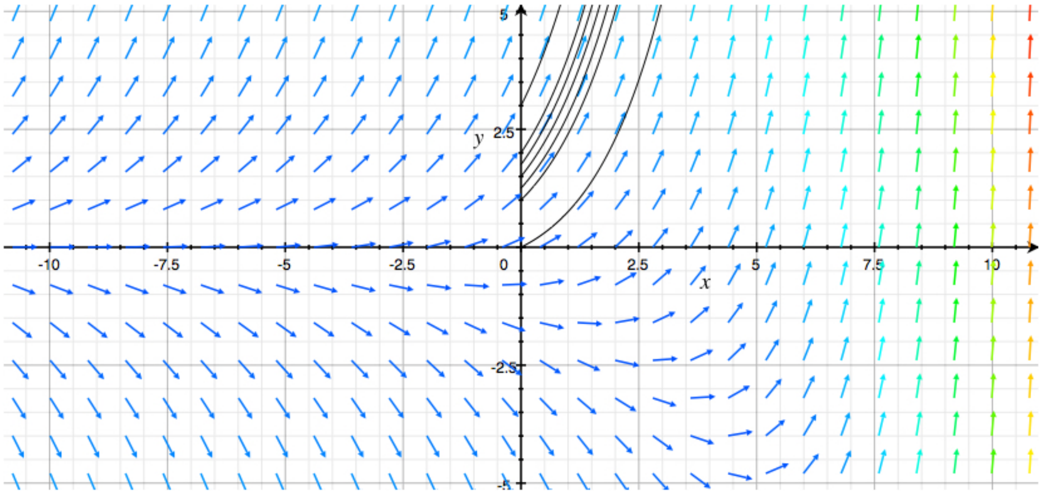

Figure 2.3: Direction field for Chapter 2.1, Problem 22.

$$y(t) = e^{t/2}[-3e^{-t/6} + C] \tag{2.8}$$
$$a = y(0) \tag{2.9}$$
$$= -3 + C \tag{2.10}$$
$$a + 3 = C \tag{2.11}$$
$$y(t) = Ce^{t/2} - 3e^{t/3} \tag{2.12}$$

The $e^{t/2}$ term will dominate as t gets large. If $a > -3$ then $\lim_{t\to\infty} y(t) = +\infty$; otherwise $\lim_{t\to\infty} y(t) = -\infty$.

Chapter 2.2

2.2.8

$$\frac{dy}{dx} = \frac{x^2}{1+y^2} \tag{2.13}$$
$$(1+y^2)\,dy = x^2\,dx \tag{2.14}$$
$$y + \frac{y^3}{3} = \frac{x^3}{3} + C \tag{2.15}$$
$$y^3 + 3y - x^3 = C \tag{2.16}$$

Chapter 2.3

2.3.8

$$\frac{dS}{dt} = k + rS \tag{2.17}$$
$$\frac{dS}{dt} - rS = k \tag{2.18}$$

Using methods from 2.1,

$$\mu(t) = e^{-rt} \tag{2.19}$$
$$S(t) = \frac{1}{\mu(t)} \int \mu(t) q(t) dt \tag{2.20}$$
$$= e^{rt} \int k e^{-rt} dt \tag{2.21}$$
$$= k e^{rt} \Big[-\frac{1}{r} e^{-rt} + C \Big] \tag{2.22}$$
$$= -\frac{k}{r} + Cke^{rt} \tag{2.23}$$
$$= \frac{k}{r}(e^{rt} - 1) \tag{2.24}$$

where we chose C to correspond with the "no inital capital" condition. Setting

$$S(40) = 1000000 = \frac{k}{.075}(e^{.075 \cdot 40} - 1) \tag{2.25}$$
$$\tag{2.26}$$
$$k = \$3929.68 \tag{2.27}$$

On the other hand, if $k = \$2000$, then

$$1000000 = \frac{2000}{r}(e^{40r} - 1) \tag{2.28}$$

Up to two decimal places, the required interest rate is 9.77%.

2.3.14

This DE is separable:

$$\frac{dy}{dt} = (0.5 + \sin t)\frac{y}{5} \tag{2.29}$$

$$\frac{dy}{y} = \frac{1}{5}(0.5 + \sin t)\,dt \tag{2.30}$$

$$\ln|y| = \frac{1}{5}(0.5t - \cos t) + C \tag{2.31}$$

$$|y| = C\exp(\frac{1}{5}(0.5t - \cos t)) \tag{2.32}$$

$$y = C\exp(\frac{1}{5}(0.5t - \cos t)) \tag{2.33}$$

where $C > 0$. At time $t = 0$, the population is

$$y(0) = C\exp(\frac{1}{5}(0 - 1)) = Ce^{-1/5} \tag{2.34}$$

hence $C = e^{1/5}$ based on the initial condition. The solution for τ is found by solving the equation

$$2 = \exp(\frac{1}{5}(1 - \cos\tau + 0.5\tau)) \tag{2.35}$$

For the given initial condition, $\tau \approx 6.7327$. On the other hand, from equation 2.34 we see that

$$C = y(0)e^{1/5} \tag{2.36}$$

in general; hence

$$y(t) = y(0)\exp(\frac{1}{5}(1 + 0.5t - \cos t)) \tag{2.37}$$

in general. But then the solution for τ reduces to the solution of

$$2y(0) = y(0)\exp(\frac{1}{5}(1 + 0.5\tau - \cos\tau)) \tag{2.38}$$

which, after division by non-zero $y(0)$, is identical to equation 2.35. Thus τ does not depend on $y(0)$, except in the spurious case where $y(0) = 0$ in which $y \equiv 0$ for all time.

2.3.16

Let T be the temperature of the cup; then

$$\frac{dT}{dt} = k(70 - T) \tag{2.39}$$

We solve this DE using our tools for separable equations:

$$\frac{dT}{70 - T} = k\,dt \tag{2.40}$$
$$-\int \frac{-dT}{70 - T} = -\int \frac{du}{u(T)} = \int k\,dt \tag{2.41}$$
$$-\ln|u| = kt + C \tag{2.42}$$
$$\exp(\ln|u|) = \exp(-kt + C) \tag{2.43}$$
$$u = Ce^{-kt} \tag{2.44}$$
$$200 = u(0) = C \tag{2.45}$$
$$190 = u(1) = 200e^{-k} \tag{2.46}$$
$$0.05129329439 = -\ln(.95) = k \tag{2.47}$$

So the time τ at which the coffee cup reaches 150° is

$$150 = u(\tau) = 200e^{-k\tau} \tag{2.48}$$
$$\ln(.75) = -k\tau \tag{2.49}$$
$$5.608570789 = \tau \tag{2.50}$$

i.e., about 5m36s after being poured.

2.3.20

Let $y(t)$ be the distance above the ground in metres and $g = -9.81\,m/s^2$; then

$$\frac{d^2y}{dt^2} = -g \tag{2.51}$$

whence

$$\frac{dy}{dt} = 20 - gt \tag{2.52}$$
$$y(t) = 30 + 20t - \frac{g}{2}t^2 \tag{2.53}$$

The maximum height occurs at the time that the ball's velocity reaches zero. That time τ is given by

$$\tau = \frac{20}{g} \text{ sec} \tag{2.54}$$

so the maximum height is given by

$$y(\tau) = 50.3873598 \text{ m} \tag{2.55}$$

The time at which the ball hits the ground is given by the positive root of $y(t)$, which is

$$t = 5.243833855 \text{ sec} \tag{2.56}$$

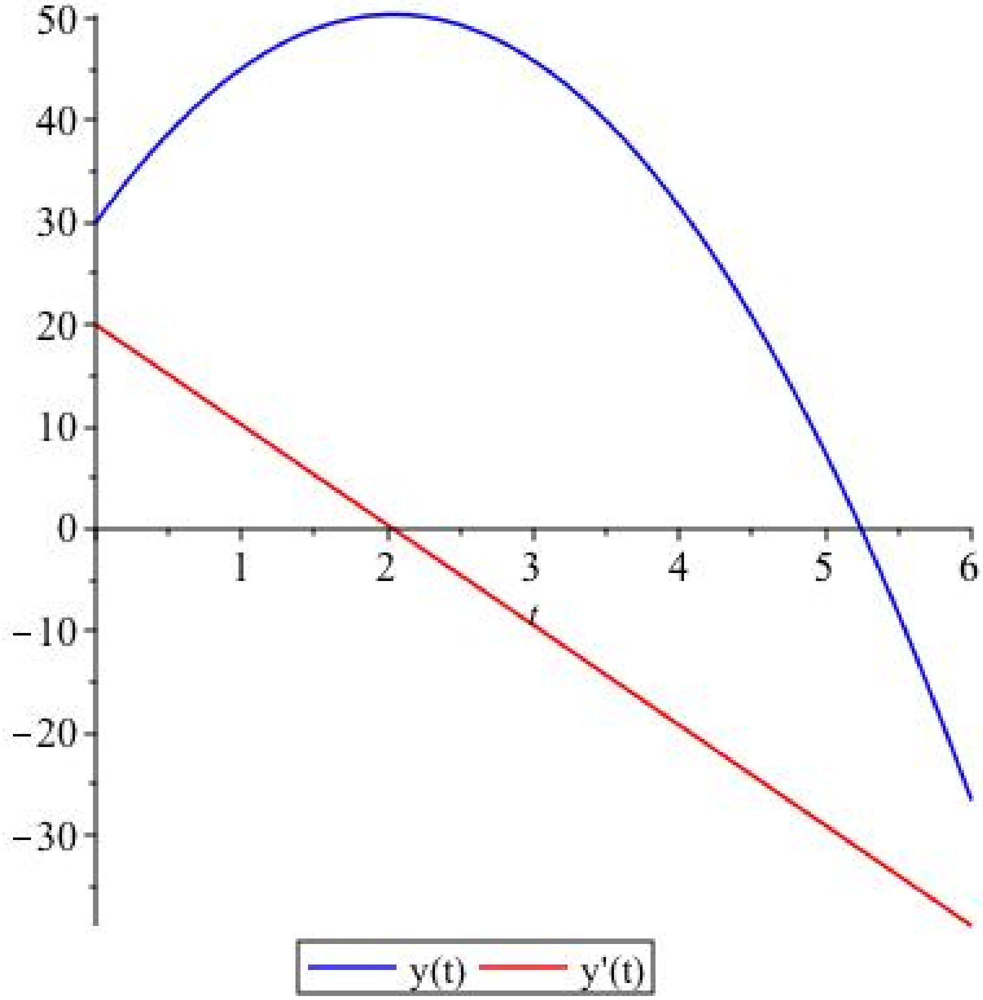

Figure 2.4: Plot of $y(t)$ and $v(t)$ for Problem 20 of Chapter 2.3.

2.3.30

Verifying the equations in (a) is straightforward. Integrating them gives

$$x(t) = (u \cos A) \cdot t + x(0) = (u \cos A) \cdot t \tag{2.57}$$

$$y(t) = -\frac{g}{2}t^2 + u \sin At + y(0) = -\frac{g}{2}t^2 + (u \sin A)t + h \tag{2.58}$$

We call τ the time at which the ball reaches, or goes over, the wall; then

$$x(\tau) = (u \cos A) \cdot \tau = L \tag{2.59}$$

and clearing the wall is given mathematically by

$$y(\tau) > H \tag{2.60}$$

The remaining calculations are the domain of algebra/precal.

Chapter 2.4

2.4.1

$$\begin{aligned} p(t) &= (\ln t)/(t-3), \\ q(t) &= (2t)/(t-3) \end{aligned}$$

The largest interval on which both p and q are continuous that contains $t = 1$ is $(0, 3)$.

2.4.7

$$\begin{aligned} f(t,y) &= \frac{t-y}{2t+5y} \\ f_y(t,y) &= \frac{(2t+5y)(-1)-(t-y)(5)}{(2t+5y)^2} \\ &= \frac{(-2t-5y)-(5t-5y)}{(2t+5y)^2} \\ &= -\frac{7t}{(2t+5y)^2} \end{aligned}$$

The hypotheses apply at any point not on the line $2t + 5y = 0$.

2.4.15

$$y' = -y^3$$

Theorem 2.4.2 applies no matter what initial conditions we set because $f(t, y)$ and f_y are continuous everywhere.

$$y^{-3}\,dy = -dt$$

If $y(t)$ is a solution and $y(c) = 0$ for any $t = c$ then $y(t) = 0$, because $y \equiv 0$ is a solution and 2.4.2 holds, forbidding any two solutions from intersecting. Hence we safely divide by y above.

$$\begin{aligned} -\frac{1}{2y^2} &= -t + C \\ \frac{1}{y^2} &= 2t + C \end{aligned}$$

If $y(0) = y_0 \neq 0$ then

$$\begin{aligned} \frac{1}{y_0^2} &= C \\ y(t) &= \pm\frac{1}{\sqrt{2t + C}} \\ &= \pm\frac{1}{\sqrt{2t + \frac{1}{y_0^2}}} \\ &= \pm\frac{1}{\frac{1}{|y_0|}\sqrt{2y_0^2 t + 1}} \\ &= \frac{y_0}{\sqrt{2y_0^2 t + 1}} \end{aligned}$$

We must avoid division by zero or taking the square root of negative numbers, hence the particular solution exists if and only if

$$\begin{aligned} 2y_0^2 t + 1 &> 0 \\ t &> -\frac{1}{2y_0^2} \end{aligned}$$

In sum, the solution to the differential equation given in 2.4.15 is

$$y(t) = \begin{cases} \frac{y_0}{\sqrt{2y_0^2 t + 1}}, & y_0 \neq 0 \\ 0, & y_0 = 0 \end{cases}$$

and the interval in which the solution exists is

$$y(t) = \begin{cases} (-\frac{1}{2y_0^2}, \infty) & y_0 \neq 0 \\ \mathbb{R}, & y_0 = 0 \end{cases}$$

Chapter 2.5

[see separate file in Resources]

Chapter 2.6

2.6.2

Not exact because $M_y = 4 \neq 2 = N_x$.

2.6.5

We must exclude the line $bx + cy = 0$; after doing so, we arrange to obtain

$$(ax + by) + (bx + cy)\frac{dy}{dx} = 0$$

This function is exact. Integrating,

$$\psi(x, y) = \frac{ax^2}{2} + bxy + h(y)$$

Partially deriving w/r to y,

$$\frac{\delta\psi}{\delta y} = bx + h'(y) = N(x, y) = bx + cy$$

which implies $h'(y) = cy$. Hence $h(y) = \frac{cy^2}{2}$ and the solution is:

$$2\psi(x, y) = ax^2 + cy^2 + 2bxy = C$$

(Note that multiplying through by 2 has no effect on the solution and is merely done to rid it of fractions so it looks nicer.)

2.6.14

We have $M_y = 1 = N_x$ (note that $N(x, y) = -4y + x$) and so the equation is exact. Integrating,

$$\psi(x, y) = 3x^3 + xy - x + h(y)$$

Partially deriving,

$$\frac{\delta\psi}{\delta y} = x + h'(y) = x - 4y = N(x, y)$$

So

$$h(y) = -2y^2$$

and the solution is

$$2x^3 + xy - x - 2y^2 = 1$$

You can find a function $y(x)$ by using the Quadratic Formula; the interval of validity will be the interval that yields a non-negative argument under the radical.

2.6.26

Rewrite the DE as

$$\mu(1 - y - e^{2x}) + \mu y' = 0$$

The DE is exact iff

$$\begin{aligned}
[\mu(1 - y - e^{2x})]_y &= \mu_x \\
\mu_y(1 - y - e^{2x}) - \mu &= \mu_x
\end{aligned}$$

If we restrict μ to functions of x only then

$$-\mu = \mu_x$$

which has solution

$$\mu(x) = e^{-x}$$

We then solve the equation multiplied by $\mu(x)$:

$$\begin{aligned}
\psi(x, y) = \int \mu(x) M(x, y)\, dx &= \int (e^{-x})(1 - y) - e^x\, dx \\
&= (e^{-x})(y - 1) - e^x + h(y) \\
\frac{\delta\psi}{\delta y} &= e^{-x} + h'(y) \\
= N(x, y) &= e^{-x} \\
h(y) &= c
\end{aligned}$$

So the solution is

$$\phi(x,y) = e^{-x}(y-1) - e^x = C$$

Multiplying through allows us to obtain a function $y = f(x)$:

$$\begin{aligned}(y-1) - e^{2x} &= Ce^x \\ y &= Ce^x + e^{2x} + 1\end{aligned}$$

Chapter 2.7

[see Chapter 8]

Chapter 2.9

2.9.14

$$\begin{aligned}
\frac{\rho-1}{\rho} + v_{n+1} &= \rho\Big(\frac{\rho-1}{\rho} + v_n\Big)\Big(\frac{1}{\rho} - v_n\Big) && (2.61)\\
\frac{\rho-1}{\rho} + v_{n+1} &= \Big(\frac{\rho-1}{\rho} + v_n\Big)\Big(1 - \rho v_n\Big) && (2.62)\\
\rho - 1 + \rho v_{n+1} &= (\rho - 1 + \rho v_n)\Big(1 - \rho v_n\Big) && (2.63)\\
&= \rho - 1 + (2-\rho)\rho v_n - \rho^2 v_n^2 && (2.64)\\
\rho v_{n+1} &= (2-\rho)\rho v_n - \rho^2 v_n^2 && (2.65)\\
v_{n+1} &= (2-\rho)v_n - \rho v_n^2 && (2.66)
\end{aligned}$$