Note. The numbering of equations is different in these solutions than on the original exam.

Question 1 [25 pts]

Find a solution $y(t)$ of the equation

$$\frac{dy}{dt} - \frac{2t}{1+t^2}\, y = 3 \tag{1}$$

subject to the initial condition $y(0) = 3$.

Solution

The integrating factor is [**3pts**]

$$\mu(t) = e^{\int p(s)\, ds} \tag{2}$$

And [**3 pts**]

$$\begin{aligned} \int p(s)\, ds &= -\int \frac{2s\, ds}{1+s^2} && (3)\\ &= -\int \frac{du}{u} && (4)\\ &= -\ln|u| && (5)\\ &= -\ln(1+t^2) && (6) \end{aligned}$$

Hence [**3 pts**] $\mu(t) = \frac{1}{1+t^2}$. The solution to a first-order linear equation is [**5 pts**]

$$y(t) = \frac{1}{\mu(t)} \int \mu(s) q(s)\, ds \tag{7}$$

where, in this case, $q(s) = 3$. So

$$\begin{aligned} y(t) &= \frac{1}{\mu(t)} \int \mu(s) q(s)\, ds && (8)\\ &= (1+t^2) \int \frac{3\, ds}{1+s^2} && (9)\\ &= (1+t^2)\big[3 \cdot \arctan(t) + C\big] && (10) \end{aligned}$$

The initial condition gives us that

$$\begin{align} 3 = y(0) &= (1 + 0^2)\big[3 \cdot \arctan(0) + C\big] \tag{11} \\ &= C \tag{12} \end{align}$$

Hence ([**11 pts**] if entirely correct, [**6 pts**] if off by const or just gives g.s.)

$$\boxed{y(t) = 3(1 + t^2)(1 + \arctan(t))} \tag{13}$$

Question 2 [20 pts]

The populations of two species, one predator and one prey, are modelled over time by $x(t)$ and $y(t)$. These functions obey the system of differential equations

$$\frac{dx}{dt} = x(-60 + 3y) \tag{14}$$
$$\frac{dy}{dt} = y(30 - x) \tag{15}$$

(a) Decide which of the functions represents the predator population and which represents the prey; justify both choices (*i.e.*, do not just write, *e.g.*, "process of elimination" for the second choice).

(b) Find all critical points of this system.

Solution

(a) $x(t)$ gives the predator population, $y(t)$ gives the prey population.

- In the absence of prey (*i.e.*, if $y = 0$), the predators will die out at a rate proportional to their population at a given moment (hence the negative coefficient in from of x in Equation 14.
- On the other hand, in the absence of predators (*i.e.*, if $x = 0$), the prey will grow at a rate proportional to their population at a given moment (hence the positive coefficient in front of x in Equation 15).
- Prey are killed, and predators are born, at a rate proportional to the number of encounters between the populations, hence the positive coefficient in front of xy for Equation 14 and the negative in front of 15.

(b) We first set the rhs of Equation 14 equal to zero to obtain

$$x = 0 \tag{16}$$
$$-60 + 3y = 0 \tag{17}$$

as possible equations that a critical point could satisfy.

Starting with the case $x = 0$, we reason that the critical point must also cause the rhs of equation 15 to vanish, *i.e.*, cause

$$y(30 - 0) = 0 \tag{18}$$

Only the origin satisfies Equation 18 as well, so $\boxed{(0, 0)}$ is a critical point of the system.

Now consider the case corresponding to the equation $-60 + 3y = 0$. Only $y = 20$ satisfies this equation, yielding a second equation of $20(30 - x) = 0$, whence a critical point of $\boxed{(30, 20)}$.

- **part (a): 4 points for correct choice of predator/prey (-2 if one of the derivatives was chosen as the function**
- **part (a): 8 points for correct explanation**
- **part (b): 4 points for each critical point**
- **part (b): minus 4 points for incorrect critical points (but part (b) cannot be scored less than zero)**

Question 3 [20 pts]

NOTE. (For purposes of this question, *stable* will be understood to mean *stable but not asymptotically stable.*)

Find, and classify as *stable*, *asymptotically stable*, or *unstable*, all critical points of the following system:

$$\frac{dx}{dt} = xy \tag{19}$$

$$\frac{dy}{dt} = x^2 + y^2 - 1 \tag{20}$$

Solution

By the same reasoning as in Question 2, the critical points can be shown to be

$$\boxed{(0, \pm 1),\ (\pm 1, 0)} \tag{21}$$

The Jacobian matrix for this system is

$$\begin{pmatrix} y & x \\ 2x & 2y \end{pmatrix} \tag{22}$$

which, evaluated at the critical points, comes out to

$$\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 2 \end{pmatrix} \tag{23}$$

if $x = 0$, or

$$\begin{pmatrix} 0 & \pm 1 \\ \pm 2 & 0 \end{pmatrix} \tag{24}$$

if $y = 0$.

Equation 23 yields two distinct eigenvalues; if the eigenvalues are positive then the critical point is **unstable**, and if they are **negative** then the critical point is **asymptotically stable**.

Equation 24 yields a characteristic equation of $\lambda^2 - 2$ regardless of the sign of the entries of the matrix, a polynomial with roots $\pm\sqrt{2}$. Thus the corresponding critical points are saddle points, which are **unstable**.

There are four critical points.

- finding each one is worth a point
- finding the Jacobian of each is worth two points per critical point
- giving the correct stability based on the Jacobian (with correct rationale) is worth two points per critical point
- Incorrect critical points receive a penalty of five points (for a minimum total question score of zero) and are not eligible for partial credit.

Question 4 [25 pts] Draw a reasonably accurate phase portrait for

$$\mathbf{u}' = \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix} \mathbf{u} \tag{25}$$

Solution

The characteristic polynomial [**5 pts**] is

$$\begin{vmatrix} 1-\lambda & 2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - (2)(-2) \tag{26}$$

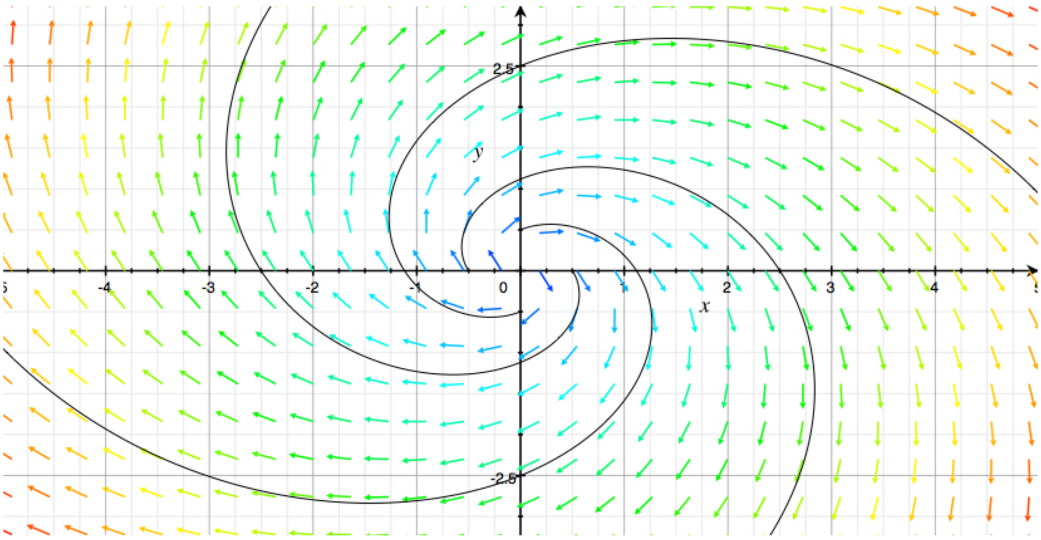
$$= (1-\lambda)^2 + 4 \tag{27}$$

$$= \lambda^2 - 2\lambda + 5 \tag{28}$$

By the quadratic formula, the roots for this equation [**5 pts**] are

$$\lambda = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \frac{2 \pm 4i}{2} = 1 \pm 2i \tag{29}$$

The positive real part of this equation indicates that we have a **spiral source (*)**; testing vectors shows that the phase portrait should spin **clockwise (*)**. A portrait with representative curves like the below (plus arrows on the curves in the direction of time, as on the direction field) is worth full credit **(*)**:



(5 pts for each asterisked item. Because this question is so specifically targeted, grading stops if any one of the non-asterisked items in sequence is wrong.)

Question 5 [10 pts]

Find the solution $y(t)$ of

$$\frac{dy}{dt} = \frac{1}{y} \tag{30}$$

that passes through the point $(0, 1)$ on the ty-axis.

Solution

Equation 30 is separable into

$$y\,dy = \; dt \tag{31}$$

which we integrate

$$\frac{1}{2}y^2 = t + C \tag{32}$$

and substitute constants into

$$\frac{1}{2} = C \tag{33}$$

$$\frac{1}{2}y^2 = t + \frac{1}{2} \tag{34}$$

$$y^2 = 2t + 1 \tag{35}$$

to obtain

$$y^2 = \pm\sqrt{2t+1} \tag{36}$$

We choose

$$\tag{37}$$

$$\boxed{y(t) = \sqrt{2t+1}} \tag{38}$$

because $y > 1$ at $t = 0$.

Each of the following is worth two points:

- separation (31)

- integration (32)
- finding C (33)
- final answer

Separation **must** be correct for any credit.

END OF PART 1

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IMPORTANT REMINDER: Be sure to finish Problems 1-5 before time is called for Part 1; you will be unable to return to Part 1 once Part 2 begins.

Question 6 [20 pts]

Find one particular solution $y(t)$ of

$$y''(t) - 2y'(t) + y(t) = e^t \tag{39}$$

Solution

Call $L[y]$ the linear differential operator with respect to time such that Equation 39 is identically written

$$L[y] = e^t \tag{40}$$

Then $L[y] = 0$ has characteristic polynomial

$$\lambda^2 - 2\lambda + \lambda \tag{41}$$

which has root 1 with algebraic multiplicity 2. Hence the general solution y_h of $L[y] = 0$ is

$$y_h = (at + b)e^t \tag{42}$$

A particular solution to Equation 39 cannot solve the corresponding homogeneous equation; hence, we must use the method of undetermined coefficients with a guess of the form

$$y_p = ct^2 e^t \tag{43}$$

Taking derivatives, we obtain

$$\begin{array}{rcrcrcr} y_p & = & ct^2e^t & & & & \\ -2y_p' & = & -2ct^2e^t & + & -4cte^t & & \\ y_p'' & = & ct^2e^t & + & 4cte^t & + & 2ce^t \end{array} \tag{44}$$

whence

$$c = \frac{1}{2} \tag{45}$$

and we obtain an answer of $\boxed{y = \frac{1}{2}t^2e^t}$

The coefficient, the final answer, and each derivative of y_p (including y_p itself) are worth 4 pts apiece. No credit for y_p if solutions to the homogeneous equation are included. Guess MUST either be t^2e^t or that plus solutions to the homogeneous equation for any partial credit for problem to be possible.

Question 7 [25 pts]

A certain spring's motion is characterized by the differential equation

$$y''(t) + 2y'(t) + 4y(t) = F(t) \tag{46}$$

where $F(t)$ is the forced vibration due to some outside influence.

Give the general solution of $y(t)$ in the case in which $F(t) = 4\sin(2t)$ and the spring starts at rest at position 0.

Solution

The homogeneous solution has characteristic polynomial with roots

$$\lambda = \frac{-2 \pm \sqrt{4 - 4 \cdot 1 \cdot 4}}{2} = -1 \pm i\sqrt{3} \tag{47}$$

whence the general solution of the homogeneous differential equation is

$$y_h = e^{-t}(a\cos(\sqrt{3}t) + b\sin(\sqrt{3}t)) \tag{48}$$

Using complexification, we seek a solution to the equation

$$u'' + 2u' + 4u = 4e^{i2t} \tag{49}$$

by making a guess of the form $u_p := ce^{i2t}$. Taking derivatives, we obtain

$$\begin{array}{rcll} 4u & = & (\ \ 4 + 0i) & ce^{i2t} \\ 2u' & = & (\ \ 0 + 4i) & ce^{i2t} \\ u'' & = & (-4 + 0i) & ce^{i2t} \end{array} \tag{50}$$

Hence, $c = -i$ and

$$u_p = -ie^{i2t} = -i(\cos(2t) + i\sin(2t)) = -i\cos(2t) + ... \tag{51}$$

We only care about the imaginary parts of u_p since $F_p = 4\sin(2t)$, so $y_p = -cos(2t)$. The general solution of the motion of the spring is thus

$$y(t) = e^{-t}(a\cos(\sqrt{3}t) + b\sin(\sqrt{3}t)) - \cos(2t) \tag{52}$$

whence

$$y(t) = e^{-t}(a\cos(\sqrt{3}t) + b\sin(\sqrt{3}t)) - \cos(2t) \tag{53}$$

Setting $y(0) = 0 = y'(0)$,

$$y(0) = 0 = \quad a - 1 \tag{54}$$

$$y'(0) = 0 = -a + b\sqrt{3} \tag{55}$$

whence

$$\boxed{y(t) = e^{-t}\Big(\cos(\sqrt{3}t) + \frac{\sqrt{3}}{3}\sin(\sqrt{3}t)\Big) - \cos(2t)} \tag{56}$$

- **5 pts** homogeneous solution
- **10 pts** particular solution (holistic, see below):
  - 10 pts: entirely correct
  - 9 pts: correct but for trivial mistake
  - 8 pts: high pass
  - 7 pts: low pass
  - 5 pts: fail
  - 0 pts: completely/mostly wrong
- **10 pts** derivative of $y(t)$ and substitution of ICs to obtain final answer

Question 8 [20 pts]

Determine the constant a such that the equation

$$(ye^{ty} - 2t + ay^2) + (te^{ty} + at^2)\frac{dy}{dt} = 0 \tag{57}$$

is exact in any given rectangle, substitute that value of a into the equation, and then use the methods for exact equations to find a solution for the equation in the form of a function $y(t)$ (since this is a non-linear equation, your solution need not be exhaustive).

Solution

Setting

$$M(t, y) := ye^{ty} - 2t + ay^2 \tag{58}$$
$$N(t, y) := te^{ty} + at^2 \tag{59}$$

it is clear that M, N, and their partial derivatives are everywhere continuous. Hence, the equation is exact in a given rectangle if and only if $\frac{\delta M}{\delta y} = \frac{\delta N}{\delta t}$ within the rectangle. We have that

$$\frac{\delta M}{\delta y} := tye^{ty} + e^{ty} + 2ay \tag{60}$$
$$\frac{\delta N}{\delta t} := tye^{ty} + e^{ty} + 2at \tag{61}$$

and these two are equal if and only if $2ay = 2at$ – *i.e.*, if $a = 0$ or $y = t$. Since $y = t$ is a line, the only way both conditions can be satisfied in an entire rectangle is if $a = 0$.

Setting $a = 0$, we integrate to obtain $\psi(t, y)$:

$$\psi(t, y) = \int M(t, y)\partial t \tag{62}$$
$$= \int (ye^{ty} - 2t)\partial t \tag{63}$$
$$= e^{ty} - t^2 + h(y) \tag{64}$$

Now since we posit that $N = \frac{\delta \psi}{\delta y}$:

$$\frac{\delta \psi}{\delta y} = te^{ty} + h'(y) \tag{65}$$
$$N(t, y) = te^{ty} \tag{66}$$

So $h(y) = C$. Hence we find a solution of

$$e^{ty} - t^2 = C \tag{67}$$
$$e^{ty} = C + t^2 \tag{68}$$
$$ty = \ln(C + t^2) \tag{69}$$

resulting in a solution of

$$\boxed{y = \frac{1}{t}\ln(C + t^2)} \tag{70}$$

when $t > 0$ and appropriate initial conditions are given.

- **2 points:**
  - M_y
  - N_x
  - $a = 0$ (No points after Equation 61 if this point is not obtained)
  - $y(t)$
- **3 points:**
  - Exact iff $M_y = N_x$
  - Integrate to obtain $\phi(t, y)$
  - Derive to obtain $h'(y)$
  - General (implicit) solution

Question 9 [20 pts] Draw a reasonably accurate phase portrait for

$$\mathbf{u}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \mathbf{u}$$

Solution

The characteristic polynomial is

$$\begin{aligned} \begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} &= (1-\lambda)^2 - (-2)(-2) && (71) \\ &= (1-\lambda)^2 - 4 && (72) \\ &= \lambda^2 - 2\lambda - 3 && (73) \end{aligned}$$

The eigenvalues are $\lambda = -1, 3$, meaning that we have a saddle point at the origin. The eigenvectors for $\lambda = -1$ and $\lambda = 3$ are given by solutions to

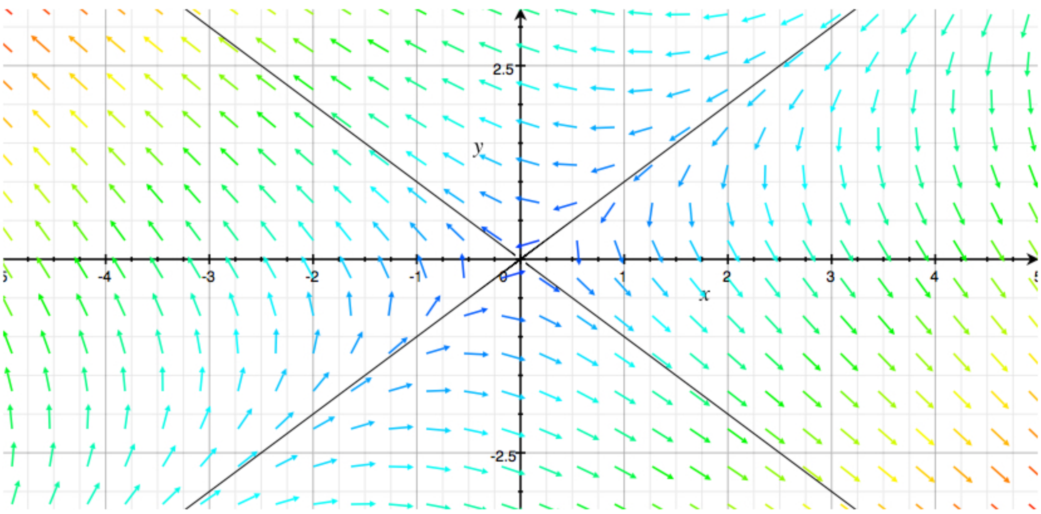
$$\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \mathbf{v} = 0 \tag{74}$$

and

$$\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \mathbf{v} = 0 \tag{75}$$

respectively; these solutions are, respectively, $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

The portrait resembles the one below. You should draw straight line solutions with appropriate slopes and representative curves with appropriate end behavior, with arrows matching the direction field you see below:



2 pts for the CP, 2 pts for the roots, 2 pts for each of the kinds of solutions on Quiz 9 (must have right arrows and slope/ e.b.)

Question 10 [15 pts]

Circle **T** (True) or **F** (False) for each of the following statements (**true** means true in all circumstances). No justification is necessary, but make sure that your choice can be clearly made out.

(There is no penalty for guessing on this question.)

1. (True, **False**) If a matrix A has n distinct eigenvalues then it must have n linearly independent eigenvectors.

2. (True, **False**) The only solution to the matrix equation $A\mathbf{v} = 0$ is $\mathbf{v} = 0$.

   By Rank-Nullity, this statement is true if and only if A is invertible.

3. (True, **False**) Damped springs may return to the equilibrium position (*i.e.*, $y = 0$ if position is modelled by $y(t)$) at most once.

   The statement is false for underdamped springs.

4. (True, **False**) When A is a 2×2 matrix with eigenvalue λ of algebraic multiplicity 2, the general solution of $\mathbf{u}' = A\mathbf{u}$ is given by

   $$\mathbf{u}(t) = c_1\mathbf{v} + c_2(t \cdot \mathbf{v} + \mathbf{w})$$

   where $\mathbf{w}$ is determined by the equation $\mathbf{w} = (A - \lambda I)\mathbf{v}$.

   This solution is the general solution exactly when A lacks two linearly independent eigenvectors. It does NOT *work when there are two of them.*

5. (**True**, False) There is exactly one solution of

   $$\frac{dy}{dt} = y^2$$

   that passes through the origin. *The stated problem is a first-order non-linear initial value problem with an initial condition, and y^2 and $2y$ (its partial derivative w/r to y) are continuous everywhere in the ty-plane. Hence, by Theorem 2.4.2, a unique solution exists.*

END OF PART 2

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