Question 1 [20 pts total] Let $W(f,g)$ signify the Wronskian of the functions f and g.

(a) Find a function $u(t)$ such that $W(t^2, u(t)) = 1$ for all t.

Solution We set up the following matrix equation:

$$\begin{vmatrix} t^2 & u \\ 2t & u' \end{vmatrix} = t^2u' - 2tu = 1 \tag{1}$$

correct equation (3pts) (1pt if correct matrix only)

Rearranging, we obtain the first-order linear equation

$$u' - \frac{2}{t}u = \frac{1}{t^2} \tag{2}$$

this idea (2 pts)

which has integrating factor

$$\mu(t) = \frac{1}{t^2} \tag{3}$$

and thus solution

$$y(t) = t^2 \int s^{-4}\, ds = t^2[-\frac{1}{3}t^{-3} + c] \tag{4}$$

The solution $\boxed{y(t) = -\frac{1}{3t}}$ works fine. **Absolutely correct answer (3pts)**

$$\tag{5}$$

(b) Show that $W(t, f(t)) = 0$ for all t *exactly* when $f(t) = ct$.

Solution **Equation (2pts) + Matrix calculations (1pt)**

$$\begin{vmatrix} t & ct \\ 1 & c \end{vmatrix} = t \cdot \begin{vmatrix} 1 & c \\ 1 & c \end{vmatrix} = t \cdot 0 = 0 \tag{6}$$

If u is such that $W(t,u) = 0$ then

$$\begin{vmatrix} t & u \\ 1 & u' \end{vmatrix} = tu' - u = 0 \tag{7}$$

Solving this FOL differential equation gives $u(t) = ct$.

This part (3pts)

$$\tag{8}$$

(c) Show that $W(f(t), c \cdot f(t)) = 0$ for all t for any constant c and function $f(t)$.

Solution The work is mostly identical to the work in (b): replace t by $f(t)$ and 1 by $f'(t)$ in the matrix to obtain a determinant of $f(t)f'(t) \cdot 0 = 0$.

Like part (b) w/ doubled point values

Question 2 [20 pts] Find the solution to the differential equation

$$xy\cos(xy) + x^2\cos(xy)\cdot y' = -\sin(xy)$$

that passes through the point $(2, \frac{\pi}{4})$.

Solution

This equation can be rearranged to the exact equation

4pts

$$(xy\cos(xy) + \sin(xy)) + x^2\cos(xy)\cdot y' = 0$$

Indeed, observe that

4pts (2pts for equality w/ wrong derivatives)

$$2x\cos(xy) - x^2 y\sin(xy) = M_y = N_x$$

We calculate

— No credit for anything beyond this point w/o M_y = N_x —

$$\psi(x,y) = \int N_x \partial y = \int x^y \cos(xy)\partial y = x\sin(xy) + h(x)$$

2pts

Hence

$$\psi_x = \frac{\delta}{\delta x}(x\sin(xy) + h(x)) = \sin(xy) + xy\cos(xy) + h'(x) = M(x,y)$$ **2pts**

Since $h'(x) = 0$, we have $h(x) = c$ and hence that the solution is **2pts**

$$\psi(x,y) = x\sin(xy) = C$$ **2pts**

On the other hand, the initial conditions give that

$$\psi(x,y) = 2\sin(2\cdot\frac{\pi}{4}) = C = 2$$ **2pts**

So the solution is

$$\boxed{x\sin(xy) = 2}$$ **2pts**

Question 3 [20 pts] Find all solutions of the autonomous differential equation

$$y' = y^3$$

2pts per yellow, 4pts per green

and indicate the interval of validity of each.

Solution First of all, $y = 0$ is a constant solution that, by Thm 2.4.2, is not intersected by any other solution; it is therefore safe to divide by y^3 and solve:

$$y^{-3}\,dy = dt \tag{9}$$

$$-\frac{1}{2}y^{-2} = t + C \tag{10}$$

$$y^{-2} = -2t + C \tag{11}$$

$$y^2 = \frac{1}{C - 2t} \tag{12}$$

$$y = \pm\frac{1}{\sqrt{C - 2t}} \tag{13}$$

We must avoid zero denominators or negative arguments to radicals; both conditions can be avoided by requiring $C - 2t \geq 0$:

$$C - 2t \geq 0 \tag{14}$$

$$\frac{C}{2} \geq t \tag{15}$$

And so the solutions to the differential equation are the constant zero solution (valid for all time) or

$$y(t) = \pm\frac{1}{\sqrt{C - 2t}}, \quad t \in (-\infty, \frac{C}{2})$$

Question 4a **2pts per yellow, 3pts per blue**

(i) [**8 pts**] Solve the homogeneous equation

$$y'' + 2y' + 2y = 0$$

The characteristic polynomial is

$$\lambda^2 + 2\lambda + 2$$

with roots $\lambda = -1 \pm i$. Hence the solution is

$$\boxed{y(t) = e^{-t}(A\cos(t) + B\sin(t))}$$

(ii) [**8 pts**] Solve the homogeneous equation

$$y'' - 5y' + 6y = 0$$

The characteristic polynomial is

$$\lambda^2 - 5\lambda + 6$$

with roots $\lambda = 2, 3$. Hence the solution is

$$\boxed{y(t) = Ae^{2t} + Be^{3t}}$$

Question 4b [24 pts]

Find the general solution for the non-homogeneous equation

$$y'' - 5y' + 6y = 6 + te^t + \sin(2t)$$

You may cite your answers from Question 4a to answer this question.

We break the problem into three subproblems:

Three pts / yellow, six pts / green

(1) $y_1'' - 5y_1' + 6y_1 = 6$

This equation has a constant solution of $y_1 = 1$.

(2) $y_2'' - 5y_2' + 6y_2 = te^t$

Guess $y_2 = (at + b)e^t$; then

$$\begin{aligned} 6y_2 &= e^t(6at + 6b) && (16) \\ -5y_2' &= e^t(-5at - 5a - 5b) && (17) \\ y_2'' &= e^t(at + 2a + b) && (18) \\ te^t &= e^t(2at - 3a + 2b) && (19) \\ a &= \frac{1}{2} && (20) \\ 0 &= -\frac{3}{2} + 2b && (21) \\ b &= \frac{3}{4} && (22) \end{aligned}$$

So $y_2(t) = \frac{1}{4}(2t + 3)e^t$.

(3) $y_3'' - 5y_3' + 6y_3 = \sin(2t)$

Complexify to

$$u'' - 5u' + 6u = e^{i2t} \qquad (23)$$

and guess $y = ce^{i2t}$:

$$
\begin{aligned}
6u &= 6ce^{i2t} && (24)\\
-5u' &= -10ice^{i2t} && (25)\\
u'' &= -4ce^{i2t} && (26)\\
e^{i2t} &= (2-10i)ce^{i2t} && (27)\\
1 &= (2-10i)c && (28)\\
\frac{2+10i}{104} &= c && (29)\\
\frac{1}{52}(1+5i) &= c && (30)\\
u(t) &= \frac{1}{52}(1+5i)e^{i2t} && (31)\\
&= \frac{1}{52}(1+5i)(\cos(2t) + i\sin(2t)) && (32)
\end{aligned}
$$

We take the imaginary part of u since $g(t)$ contained the sine function:

$$
y_3(t) = \frac{1}{52}\sin(2t) + \frac{5}{52}\cos(2t) \tag{33}
$$

And so the general solution to the non-homogeneous equation is

$$
\begin{aligned}
y(t) &= y_h + y_p && (34)\\
&= \boxed{Ae^{2t} + Be^{3t} + 1 + \frac{1}{4}(2t+3)e^{t} + \frac{1}{52}\sin(2t) + \frac{5}{52}\cos(2t)} && (35)
\end{aligned}
$$