The logic of ϵ-δ proofs.

1. Consider the fact

$$\lim_{x\to 4} 2x + 1 = 9$$

   (a) Write out the definition of this statement using the ϵ-δ definition of the limit.

   (b) Take the expression $|2x + 1 - 9|$ and simplify it so it has a factor of $|x - 4|$ in it.

   (c) If I want to have $|2x + 1 - 9| = 0.01$, what value do I need to pick for $|x - 4|$? Notice that if I want $|2x + 1 - 9| < 0.01$, I can do this by taking $|x - 4|$ less than this value.

   (d) If I want to have $|2x + 1 - 9| = 0.001$, what value do I need to pick for $|x - 4|$? Notice that if I want $|2x + 1 - 9| < 0.001$, I can do this by taking $|x - 4|$ less than this value.

   (e) If I want to have $|2x + 1 - 9| = \epsilon$, what value do I need to pick for $|x - 4|$? Notice that if I want $|2x + 1 - 9| < \epsilon$, I can do this by taking $|x - 4|$ less than this value.

   (f) Now, set δ to be the value you calculated in the last step. Do the calculations from the previous step in reverse to show that if $|x - 4| < \delta$, then $|2x + 1 - 9| < \epsilon$. Idea: If $|x - 4| < \delta$, then we can plug that into the rewritten version of $|2x + 1 - 9|$ to get the bound that we want.

2. Now, consider the following limit.

$$\lim_{x\to 3} \frac{1}{x} = \frac{1}{3}$$

   (a) Write out the ϵ-δ definition of this limit.

   (b) Take the expression $\left|\frac{1}{x} - \frac{1}{3}\right|$ and rewrite it so that it has a factor of $|x - 3|$.

   (c) This problem is more complicated than the previous one because the extra factor attached to the $|x - 3|$ is not a constant, but depends on x. But we can handle this. We know that we want x to be close to 3 in this limit. If $|x - 3| \leq 1$, then what are the largest and smallest possible values of x?

   (d) With those largest and smallest values of x, what are the largest and smallest values of this extra factor?

   (e) What these last two steps have shown is that if $|x - 3| < 1$, then this extra factor has an upper bound. So, as long as we make sure $\delta < 1$, that upper bound works whenever $|x - 3| < \delta$. Assume that we have this, and replace the extra factor by that upper bound to get an expression of the form

$$\left|\frac{1}{x} - \frac{1}{3}\right| < C|x - 3|$$

where C is the bound you found in the previous part.

(f) This problem now looks a lot like the first example. Use the same techniques to figure out what we would need to make δ in terms of ϵ to make this definition work again.

(g) Before we go through the problem with defining δ and moving forward, we need to take into account the fact that we needed $\delta < 1$ for the non-constant factor to be bounded. Thus, we need both $\delta < 1$ and $\delta <$ some bound from the previous part. Write an expression using min in order to accomplish both of these.

(h) Now we can pick δ to be less than this value. Work though the calculations in the reverse order to get that

$$\left|\frac{1}{x} - \frac{1}{3}\right| < \epsilon.$$

Make sure you see where each of the two assumptions on δ come into play to give this result.