Consider the functions

$$f(x) = \begin{cases} x & x \geq 1 \\ 2 - x & x < 1 \end{cases} \qquad g(x) = \begin{cases} x + 2 & x \geq 1 \\ 4 - x & x < 1 \end{cases} \qquad h(x) = \begin{cases} x - 2 & x \geq 3 \\ 4 - x & x < 3 \end{cases}$$

(a) Sketch the graphs of these three functions and label them f, g, and h appropriately.

(b) A student looked at these three functions and wrote the assertions (i) and (ii) below

(i) $g(x) = 2 + f(x)$ for all x

(ii) $h(x) = f(2 + x)$ for all x.

Exactly one of these assertions is incorrect. Which one is wrong? What change would you make in order to correct this?

(c) Explain why this particular change is correct for this situation. Your answer should be a general explanation about how functions behave under these types of operations (vertical and horizontal shifts) and how those are transcribed into function notation. You can use the general rules from the textbook in order to do this.

(d) Part (a) highlights a common misconception about the role of addition in function transformations. The goal of Part (c) is to identify a common misconception about the role of multiplication in function transformations. Start by constructing functions f, g, and h so that

(iii) $g(x) = 2f(x)$ for all x

(iv) $h(x) = f(2x)$ for all x

and draw their graphs. Next, imagine that you show these graphs to some friends and ask them how to obtain g and h by transforming f. Which transformation are they more likely to identify correctly, (iii) or (iv)? For the one that is harder to identify, what wrong answer(s) would you expect to get from your friends? The information on page 9 of your textbook may be helpful.

(e) While you were trying to cook up functions f, g, and h for Part (c), a friend noticed you writing down relations (iii) and (iv) and exclaimed "Oh, didn't you notice that $g = h$? Just look at the example $f(x) = x$. You get $g(x) = 2x$ and $h(x) = 2x$, which are the same!" Convince your friend that this is not always the case by showing examples where g and h are different and by explaining the difference between the transformations found in (iii) and (iv).