Teaching Portfolio - Teaching Assistant Supplement

Rutgers University

Matthew Charnley

October 2018

Contents

Chapter 1

Introduction

This document provides an in-depth summary of my experiences of the classes for which I was a teaching assistant during my time at Rutgers. It includes my summary of each of these classes, a sample of the materials that I created for these classes, and the student feedback from these classes.

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EDUCATION

Doctor of Mathematics,
Rutgers University, Piscataway, NJ
May 2019 (expected)
Concentration: Partial Differential Equations
Advisor: Michael Vogelius

Bachelor of Science, Mathematics and Chemical Engineering
University of Notre Dame, Notre Dame, IN
May 2013

HONORS AND AWARDS

Spring 2016 - Invitation to RASTL
Spring 2018 - TA Teaching Excellence Award, Rutgers University Math Department
Fall 2018 - TAPCast, Invited Guest, Season 1 Episode 4: Active Learning, with Sandra Medina (Spanish). Released October 15, 2018. `http://go.rutgers.edu/y87h66ik`

TEACHING EXPERIENCE

Fall 2018 - Introduction to Probability
Summer 2018 - Differential Equations
Summer 2017 - Ordinary Differential Equations
Summer 2016 - Introduction to Probability
Summer 2015 - Calculus 3

TEACHING ASSISTANT EXPERIENCE

Spring 2018 - Numerical Methods in PDEs/Numerical Analysis 2
Fall 2017 - Advanced Calculus for Engineers
Fall 2016 - Graduate Real and Complex Analysis
Spring 2016 - Numerical Methods in PDEs
Spring and Fall 2015 - Calculus 3
Fall 2014 - Calculus 1

PROFESSIONAL DEVELOPMENT

Rutgers Academy for the Scholarship of Teaching and Learning (RASTL) — Fall 2016 - Spring 2019
Rutgers University, New Brunswick, NJ
- Met with graduate students from various departments at Rutgers to discuss pedagogy and other issues with teaching.
- Led workshops for the group on ‘Engaging Students and Managing Discussions’ and ‘Classroom Expectations’.

Rutgers TA Project — Fall 2016 - Spring 2019
Rutgers University, New Brunswick, NJ
- Ran and assisted in workshops run by the TA Project for graduate students at Rutgers interested in learning more about teaching.
- Discussed topics such as ‘Teaching a Summer Course’, ‘Teaching Non-Majors’, and ‘Providing Feedback that Matters’.
- Presented to new graduate students at TA Orientation, informing them of how the TA program in the math department works and where they can go to get extra teaching resources.

Rutgers Math Teaching Group Fall 2016 - Spring 2019
Rutgers University, Piscataway, NJ

- Planned a weekly seminar where graduate students from the math and math education departments could discuss teaching issues.
- Invited speakers from other departments at Rutgers to discuss programs that exist to support teaching.
- Participated in a semester-long discussion of the Calculus sequence at Rutgers and ways that it could be improved.

Math TA-At-Large Program Organizer Fall 2017 - Spring 2018
Rutgers University, Piscataway, NJ

- Organized the online office hours for the TA-At-Large program.
- Gathered class information from the TAs to provide to the technology staff to get the classes set up in the online system.
- Scheduled the 12 sets of office hours in the single technologically-capable office each of two semesters.
- Provided an outline to the department on how to run this program in future semesters.

MENTORING EXPERIENCE

Directed Readings Program Fall 2014 - Spring 2018
Rutgers University, Piscataway, NJ

- Mentored advanced undergraduates in independent study projects, showing them a little bit about what advanced math looks like.
- Advised students on giving a 15 minute presentation about the semester's work at the end of the project.
- Supervised 6 projects related to the Hydrogen Atom, Fourier Analysis, Functional Analysis and the δ-function, and Markov Chains.

Summer Session Head TA Summer 2016 - Summer 2018
Rutgers University, Piscataway, NJ

- Served as a peer reviewer for other graduate students teaching classes over the summer.
- Provided a formative assessment to each graduate student near the start of their summer class.
- Advised peers on how to improve their teaching before the summative review later in the summer by a full-time faculty member.

PUBLICATIONS

1. Charnley, M and M. Vogelius. In preparation.
2. Charnley, M and A. Wood. In preparation.
3. Charnley, M. "The average of a polygon is an ellipse." 2018. MAA Mathematics Magazine. (submitted)
4. Charnley, M. and A. Wood. "A linear sampling method for through-the-wall radar detection." 2017. DOI: `10.1016/j.jcp.2017.06.035`.
5. Charnley, M. and A. Wood. "Through-the-wall radar detection analysis via numerical modeling of Maxwell's equations." 2016. DOI: `10.1016/j.jcp.2016.01.039`.

EXTERNAL TALKS

1. "A Modern Approach to Gårding's Asymptotics Result." Rutgers Camden Several Complex Variables Learning Seminar. Rutgers University - Camden Campus. March 30, 2018.
2. "An energy lemma and an application to thin inhomogeneities." NYS Regional Graduate Mathematics Conference. Syracuse University. March 24, 2018.
3. "Flipped Classrooms for Higher Level Mathematics." NE RUME Conference. Montclair State University. November 11, 2017.
4. "A Linear Sampling Method for Through-the-Wall Radar Detection." Colloquium at the Air Force Institute of Technology. March 2, 2017.
5. "Numerical Simulation of Maxwell's Equations for Radar Detection Analysis." Brown Bag Seminar at the Air Force Institute of Technology. August 27, 2015.

OTHER RESEARCH

ORISE Summer Researcher — Summer 2015 - Present
Air Force Institute of Technology, Dayton, OH

- Continued work from a Master's Thesis on "Through-the-Wall" imaging.
- Developed techniques for generating data and analyzing the results numerically, implementing both existing research and new ideas.
- Presented findings at AFIT at several of their seminars.

Chapter 2

Teaching Assistant Course Summaries

2.1 Fall 2014 - Math 135

2.1.1 Class Overview

MATH 135 is Rutgers' version of Calculus for Non-Majors. This class serves students in some of the science and business majors who need Calculus for their programs, but do not need the rigor of the MATH 151 sequence. Students in these classes are also generally not able to go on to Calculus 3 (MATH 251) unless they jump to the engineering/majors version of Calculus 2 first. When I taught this class, the TA responsibilities took place in 55 minute recitation sections, consisting of around 35 students each, and all 105 of them had a single lecture twice a week. Since then, the 135 recitations have been extended to 80 minutes. but the general setup has remained the same. During the semester, I was also responsible for creating and grading the quizzes that were given each week in class.

2.1.2 Sample Course Materials

Sample materials that I created for this course can be found in Appendix A.1. Included in this section are

1. My syllabus for the recitation sessions
2. The set of practice problems that I handed out at the first recitation
3. A full solution to a quiz that was posted online for the students
4. A document sent to the students to help them review the topics for the common final exam

2.1.3 Student Feedback Summary

The full set of responses from the student surveys for this class can be found in Appendix B, at the end of this document. A few selected comments from these surveys are shown here.

- "Teacher is very friendly; he closes the distance by memorizing every single students' name after one or two classes..."
- "Matt did a great job of answering out questions, and helping us through the hard concepts. He was very relatable."
- "The TA is helpful and supportive of [finding things out and linking concepts], even when its not during his office hours I have emailed him and received a quick response with help but not a totally detailed solution – he helps you solve the problem."
- "It isn't an easy subject for me to understand, but he made it easy for me to understand."

- "He worked to make sure that every student's questions were answered to their satisfaction. He helped me achieve a deeper understanding of calculus, a topic I expected to struggle with."

2.1.4 Class Reflection

I feel like this was a great class for a first teaching assistant assignment. Since the material was Calculus 1, particularly for non-majors, I did not need to spend time reviewing the content for the class, and could spend more of my time working on how I was going to present the material. With only 55 minute class periods, I had to be particular with how I spent my time, since there was nowhere near enough time to talk about everything that I wanted to in each class. I think having these short recitations helped me learn how to prioritize the important topics within a class session. That is definitely something that has persisted throughout all of my other teaching assignments, even though all of them have had a longer amount of time in each class period.

I also think I began to discover my personality as an instructor during this class. As can be seen in the comments from students, I tend more towards the friendly side with my class. I believe strongly in the interactive side of teaching: that students learn best by doing problems, talking about them, and being told not only when their answers are wrong, but also what things they are doing correctly. I tried to emphasize this with my students this semester, and I think it went very well. Finally, this class affirmed how much I enjoy teaching at the college level, and motivated me to continue to pursue teaching positions both as a TA and as a full instructor during summer courses.

2.2 Spring 2015 - Math 251

2.2.1 Class Overview

MATH 251 is the Calculus 3 class offered at Rutgers. This class is taken by people in engineering, physics, and math majors, and follows in the standard calculus sequence. The teaching assistant duties for this class included running 80 minute recitations for three sections of the class each week, each containing around 20 students. I was also responsible for writing and grading quizzes for the class. This class also introduces students to Maple, a computer algebra system that can be used to visualize surfaces in three dimensions. They complete five labs over the course of the semester that walk them through setting up programs to look at things like curvature, Lagrange multipliers, and vector operators. As the TA, I was also in charge of helping the students complete the labs and understand how Maple works.

2.2.2 Sample Course Materials

Sample materials that I created for this course can be found in Appendix A.2. Included in this section are

1. My recitation syllabus handed out during the first week of class
2. Solutions to a sample quiz given during recitation
3. An example document on Clairaut's theorem that was distributed to the class to illustrate that the conditions of this theorem are necessary

2.2.3 Student Feedback Summary

The full set of responses from the student surveys for this class can be found in Appendix B, at the end of this document. A few selected comments from these surveys are

- "Matt has a very upbeat personality and really seems like he enjoys teaching."
- "stop being condescending."
- "The recitations were a perfect supplement to the lectures... Matt usually had a much clearer organized way of teaching and explaining things."
- "... You made the material so much easier to understand and you made calc 3 as fun as it could possibly be."

2.2.4 Class Reflection

This class was another great experience for me. I very much enjoyed the calculus 3 material, since it is more similar to what I am interested in for research. As the students were slightly more advanced and I have a background in engineering, I was able to hint at things that they would see in later classes, which I think they found useful. The recitation sections were 80 minutes long instead of 55 minutes with my first assignment, I had more time to go over material in class, but I still had to be careful with my time, since problems in calculus 3 take much longer to solve than problems in calculus 1, especially those at the 135 level. It was also my first introduction to Maple, as I had never used it before I was assigned to teach this class.

An important experience from this class for me is the fact that not everyone in a class is going to be a fan of how I teach and how I present things. I try to be more on the friendly side with my students, but sometimes my personality may not rub off on people in a positive manner. No matter what I do in the classroom, I will not be able to please everyone at all times. I can try as hard as I want, but eventually, I will have to make a decision about how things will be handled in the class and go with it. This class overall helped me to be more confident in the decisions I make with how I run a classroom, which definitely played a role later on in how I designed my classes over the summer.

2.3 Fall 2015 - Math 251

2.3.1 Class Overview

This was more or less the same class that I had been a teaching assistant for the previous Spring. The only difference is that it was an honors section of Calculus 3. This meant that all of the students in the class were members of the honors college, the class could ostensibly be taught at a higher level, and all of the sections were smaller. There were only about 50 students total between all three of my recitation sections, as opposed to the 75 I had during the Spring semester. Overall, the course material was identical to what had been covered in the previous semester, so I was covering the same material with a smaller and more involved class of students.

2.3.2 Sample Course Materials

Sample materials that I created for this course can be found in Appendix A.3. Included in this section are

1. My recitation syllabus handed out the first week of class
2. My written-up solution to part of the midterm that was given during my recitation session
3. Images of the lecture notes for the first section of extra video lectures that I recorded to make up for missed material in class

The most interesting of these is the last one: the set of lecture notes from the video lectures. This was my first endeavor into recording content online that students would be able to view on their own schedule. I also had to create the notes for these lectures, which was a great experience of itself.

2.3.3 Student Feedback Summary

The full set of responses from the student surveys for this class can be found in Appendix B, at the end of this document. A few selected comments from these surveys are

- "It cleared up anything I didn't understand in lecture. Was very clear and easy to follow."
- "I liked how Matt was always positive in class and clearly explained the material."
- "He made the material much easier to learn, which made me more motivated to listen and stay attentive in class."
- "... Matt always made himself available to help us learn more outside of the classroom. He knows exactly what he is talking about."

- “Matt was one of the best math TAs I’ve ever had. He was detailed and extremely clear with all the lessons he taught.”
- “Matt has really sparked my interest in mathematics and made multivariable calculus an easy to understand and fun course.”
- “Class was not only informative and interesting but fun as well, making me excited for calculus class each week.”

2.3.4 Class Reflection

This semester of Calculus 3 was another growing experience. With the smaller class size and the fact that the students were substantially more involved with the class, I felt like I got to know my students better over the course of the semester. This semester, I was also coming off of having taught Calculus 3 the previous summer, which was a very interesting transition. While I liked being in the teaching assistant role, being much less involved with the class was a little difficult for me. There are things that came up over the course of the semester that I would have done differently, but as the teaching assistant, I could only do so much. It was a good lesson in knowing how to fill my role in a class and knowing where the boundaries are in what I can and should do.

At the end of the semester, the lectures for my class had ended up about 4 lectures behind, skipping some material at the end of the semester. In particular, the class had missed out on Stokes’ Theorem and the Divergence Theorem, both of which are concepts that show up in later engineering classes. In order to compensate for this, I decided to do a series of online lectures during Winter break that the students could watch to learn about these concepts. I recorded the lectures from my house and posted them, along with the notes I had written up during the lecture, to my website. I don’t know if any of the students watched the lectures or learned anything from them, but I enjoyed making them. At the time, I thought it was something nice to do for my students, but in retrospect, I feel like this was a first step towards the flipped classroom I ran during the Summer of 2017. Without making these short lectures, I don’t know if that class would have happened in the form that it did.

2.4 Spring 2016 - Math 575

2.4.1 Class Overview

MATH 575 is a class on Numerical Methods in PDEs for students in the Mathematical Finance Masters Program. It is generally taken by students in their first or second year of the program. They are required to take a Numerical Analysis course in the fall semester, and then can either take Numerical Analysis 2 or Numerical Methods in PDEs during the Spring. The course covers finite difference and finite element methods, containing both theoretical exercises and programming problems. The goal is that after leaving this class, students will be able to understand how numerical methods work, and potentially be able to write programs to do so for their financial jobs in the future.

2.4.2 Sample Course Materials

Sample materials that I created for this course can be found in Appendix A.4. As mostly a homework grader, I didn't have too much of a role in generating content, outside of homework solutions. Included in this section are

1. A sample homework solution done in MATLAB
2. A sample homework solution with more conceptual type problems
3. A sample piece of Python code from another homework set
4. A document outlining some properties of variational formulations, which I noticed the students were having trouble with

2.4.3 Student Feedback

There was no student feedback provided for this course.

2.4.4 Class Reflection

For this semester, I served as a grader for the class. My responsibilities were to hold office hours, grade the weekly homework sets (including coding exercises) as well as helping to proctor the exams. The teaching assignment as a whole was not all that interesting. I got some more experience with coding in Python, which some of the later assignments involved, but not much else came out of the assignment.

2.5 Fall 2016 - Math 501

2.5.1 Class Overview

During this semester, I served as a teaching assistant for the two first year graduate analysis classes: MATH 501-Real Analysis I and MATH 503-Complex Analysis I. My responsibilities included completing the homework sets each week so that I could present them to the class, prepare problem sets consisting of exercises from the written qualifying exams, and run recitation sessions where I would go over all of these with the students. Each week, I would run a single 80 minute recitation session where I would cover material from both classes. I also provided extra assistance to the students during office hours if necessary.

2.5.2 Sample Course Materials

Sample materials that I created for this course can be found in Appendix A.5. Included in this section are

1. A problem set from the middle of the semester, containing written qualifying exam problems
2. The last problem set of the semester, containing a topic review to help them prepare for the finals

2.5.3 Student Feedback

There was no student feedback provided for this course.

2.5.4 Class Reflection

This class was a nice experience for me. I was allowed to review the material from the two first year classes and got a chance to present higher level material to other graduate students. It required a decent amount of preparatory work on my part in order to get ready for each of the sessions, but it was worth it for the experience of running this class.

One of the big things that came out of this semester was the fact that I need to have stronger boundaries with my students. Over the course of this semester, I felt like several of the students took advantage of the fact that I was always around my office and only a few feet away from the first year office to come by very frequently to ask for help. In semesters since then, I have tried to make sure that my office hours are explicitly laid out in my syllabus and tried to make myself much less available outside of those times. I am still willing to set up appointments with students, but I try not to meet with students outside of these times.

2.6 Fall 2017 - Math 421

2.6.1 Class Overview

MATH 421 at Rutgers is a class called Advanced Calculus for Engineers. It serves as the last math class that some of the engineering majors need to take for their major. The class covers Laplace Transforms, Matrix Algebra, Fourier Series, and Separable PDEs. It is an assortment of topics that the engineering department decided that they wanted their students to know. This class was also a unique assignemnt as it was a 'TA-At-Large' position. I did not have to run recitation sessions for this class, but was instead assigned to two 90-student sections of MATH 421. I held two sets of in-person office hours each week along with one set of online office hours. These online office hours were recorded and posted onto the Learning Management system so that students could view them later. The last bit of my responsibilities were to run review sessions for each of the sections before each of their midterms and the final.

2.6.2 Sample Course Materials

Sample materials that I created for this course can be found in Appendix A.6. Included in this section are

1. A worked out homework exercise that was posted to the students after they were having trouble with these types of problems
2. A description of Sturm-Liouville problems to help with something the students were not understanding correctly
3. A review sheet handed out for the second midterm
4. Images of the whiteboard that was projected to the students during one of my online office hours
5. The handout for a workshop that was run at the end of the semester to help review for the final.

The whiteboard screens provide a decent picture into what online office hours looked like. I generally had one or two students show up to each online office hours, and I feel like those students got a lot out of it. I don't know how many of them watched the office hours later, but I don't think it was that many. The last document is something that the other TA for MATH 421 and I did entirely on our own, and was beyond what we needed to do for the course. We held a workshop-type session where students from all 4 sections were able to come to work on problems with the two TAs there to help them figure things out before the final exam.

2.6.3 Student Feedback Summary

The full set of responses from the student surveys for this class can be found in Appendix B, at the end of this document. A few selected comments from these surveys are

- "Matt is the best TA I have evern had and is one of the smartest people I have even talked to. No matter how poorly you word a question, Matt somehow knows what you mean and has an answer that goes above and beyond."
- "It would have been nice to see Matt in class more because he was always so helpful..."
- "Always helpful and showed me the 'behind the scenes' of how we could get to certain conclusions, made me more interested in math."
- "Good sense of humor with a very positive attitude toward students."
- "Matt gives the most practical essence of the course."

2.6.4 Class Reflection

This class was a very interesting experience for me. The 'TA-At-Large' assignment was different than anything I had ever experienced as an instructor. I felt like I had a very minimal connection with the students in the classes throughout the semester. Up until the first review session, I only interacted with the few students that decided to attend office hours, and was feeling fairly unfulfilled with my teaching assignment. After the first review session, a few more students decided to come to office hours, since they realized I was a person who existed and could help them, and things got a little better. I also felt like I had very little impact on the class, and had to again remind myself that it was ok if I was not in control of everything, especially with this semester coming on the heels of my flipped classroom for MATH 244. I think it was in this semester that I learned how much I value the connection that can be established between an instructor and students, and how far this can go to foster learning of the material. Once I was able to actually talk with students about the material, they seemed to be able to understand it better.

At the end of this semester, the other TA for MATH 421 and I decided to run workshop sessions for the students to cover the material that was going to be on their final exam. We prepared packets for them on each of the 4 main topics that were covered in the class for them to work through. The actual sessions were 3 hours long, and there were two sessions run for each of two weeks, with each week covering two of the topics. After an entire semester of not really interacting with students, the workshops were fantastic. We had about 100 students in total (out of about 300) show up to at least one of the sessions, and it seemed like they got a lot out of it. We also very much enjoyed getting to actually help students through problems instead of them just bugging us for answers when they didn't get something right. It was also nice to be in charge of a session like this because we knew exactly what was going to be covered, as opposed to the normal behavior during the semester, where we would be almost guessing as to what was covered in class, and didn't really have any definitive information to give to students. In addition to the importance of the connection between instructor and student, I feel like these workshops reinforced my belief in collaborative learning, and how working in a group setting helps everyone involved to learn and understand everything better.

2.7 Spring 2018 - Math 575

2.7.1 Class Overview

For this semester, I was given the Masters in Math Finance assignment again. This time however, there were actual recitation sessions involved and it was connected to both the Numerical Methods in PDE (575) class and the Numerical Analysis 2 (574) class. The recitations were once a week for each class. In one of the classes (575) I was responsible for collecting homework, going over homework solutions, and giving the students a quiz. For the second class (574) I was only responsible for going over homework solutions for the assignment that had been collected the day before. I was not responsible for grading this time around, but the recitations sessions made up for that.

2.7.2 Sample Course Materials

Sample materials that I created for this course can be found in Appendix A.7. My main responsibilities for this course were creating quizzes and homework set solutions for the course. Included in this section are

1. My recitation syllabus from Math 574
2. A solution set to an optional homework set for Math 574 at the end of the semester that was posted for the students to use for review
3. My recitation syllabus from Math 575
4. A diagnostic linear algebra quiz that was given to students early in the semester
5. A sample homework set solution that was given to the students and the homework grader

2.7.3 Student Feedback

Student feedback is included in Appendix B. There aren't any comments of note to include here.

2.7.4 Class Reflection

I was not entirely thrilled with this assignment, but I tried to make the most of it. Each week, I was responsible for going over the homework solutions with the class. I would be responsible for writing up these homework solutions with basically no guidance from the instructors. In addition, because the class is largely international students, they had a strong desire to see the solutions written out in full detail, and so I was also responsible for typing, editing, and posting the solutions to the course website. As this was either material that I had barely learned 3 years ago or had never seen before, writing up these solutions was no simple task. On several occasions, I had to look up books or other resources to learn about the material before I could actually write solutions and solve the problems. It took up a very large portion of my time, and seemed on the whole like it didn't really have an impact on the students in the class.

As to the student population, it was challenging to get them to be involved in class while I was trying to run recitations. At this point, they had already turned in the homework, and so either understood the assignment and didn't need my help, or were past it and ready to move on, and didn't want to understand it. I also had issues with students trying to talk during quizzes, which was not even a problem in my basic calculus classes. This was the first time that recitations had been run for this class, so I think there is some adjustment to be made in this area, both on the end of the instructor/teaching assistant and from the students.

This semester also ended with me filling in for the instructor of the 575 for lectures while continuing to run recitations, write up homework solutions, and give quizzes. While the instructor was still preparing the lectures and problem sets, since I was not too knowledgeable about the material, it took a significant amount of time to get the lecture ready for class. The last few weeks of the semester were hectic with this addition, but I got through it, and I think teaching a master's level class for a month is an experience that will stick with me into the future. In particular, it showed me how different levels of student need to be treated differently, and somehow not in the ways I would expect from the outset. It also taught me the importance of preparing my own lecture notes for a course, because the lectures where I had enough time to somewhat prepare my own material went much smoother than the ones where I was reading straight from the professor's lectures.

Appendix A

Sample Course Materials

A.1 Fall 2014 - Math 135

This section contains the following documents from my teaching assistant assignment for Math 135:

1. My recitation syllabus handed out during the first week of class
2. A set of practice problems that were given out at the first recitation to make sure students knew the prerequisite material
3. A detailed written up solution to one of the quizzes given in recitation
4. A review sheet handed out to the students to help them prepare for the final exam

Math 135: Recitation Syllabus

Teaching Assistant: Matt Charnley
Email: charnley@math.rutgers.edu
Office: Hill 606, Busch Campus
Office Hours:

Class Times: Section 70: Tuesdays, 3:35-4:30, LSH-A121
Section 71: Tuesdays, 5:15-6:10, LSH-A121
Section 72: Tuesdays, 6:55-7:50, LSH-B110

Goal: The goal of these recitation sessions is to review and answer questions about the material discussed in lecture. They will deal with both general topic-related questions and specific questions from the homework. The recitation will generally go as follows:

1. Topic overview of what was covered in the past week of lecture.
2. Help/Solutions to questions on the homework/topics.
3. Quiz on the lecture topics from the previous week.

Quizzes: Quizzes will be given at every recitation session. Any student who is late to the recitation session will not be allowed to take the quiz that day. Quizzes will be graded and returned to you at the next recitation session. Solutions to the quizzes will be posted on Sakai after lecture on the same day the quiz is given out, therefore, quizzes are not allowed to be made up, for any reason. Out of all of the quiz grades earned throughout the semester, only the top 10 will be counted towards the final grade, allowing for quizzes to be missed in emergency circumstances without affecting the final grade.

Course Details: For more information about the course as a whole, please refer to the official syllabus given out in lecture.

Other Help: Calculus is a difficult class. You will need to put a lot of work into understanding the material to succeed in this class. It is important never to fall behind because the material builds on itself very rapidly; if you are behind one day, you will be completely lost on the next. Prof. Lieberman and I are both here to help you understand and learn the material. There are also several resources available from the university; please check out the Sakai site and the official syllabus for more information.

Recitation 1: Extra Practice

September 2, 2014

1. Factor these quadratic expressions.

a) $x^2 - 25$

b) $x^2 + 6x + 9$

c) $x^2 - 8x + 15$

d) $3x^2 + 11x - 4$

e) $x^2 + x - 21$

f) $2x^2 + 4x - 1$

2. Simplify the following rational and radical expressions.

a) $\sqrt{450}$

b) $\frac{225}{600}$

c) $\sqrt{28(x-4)^2(x-5)}$

d) $\frac{x^2-4x+4}{x^2+x-6}$

e) $\sqrt{72(x^2-4)(x^2+5x+6)}$

f) $\frac{x^2-9}{x^2+6x+9}$

3. Graph the following piecewise-defined functions.

a) $f(x) = \begin{cases} -1 & x < -1 \\ 2x - 1 & -1 \leq x < 2 \\ 1 & x \geq 2 \end{cases}$

b) $f(x) = \begin{cases} -x - 2 & x < -2 \\ x^2 & -2 \leq x < 1 \\ x - 4 & x \geq 1 \end{cases}$

c) $f(x) = \begin{cases} 2x & x < -1 \\ -x & -1 \leq x < 3 \\ (x-2)^2 & x \geq 3 \end{cases}$

d) $f(x) = \begin{cases} 1 & x < -1 \\ x + 1 & -1 \leq x < 2 \\ 2 - (x-2)^2 & x \geq 2 \end{cases}$

e) $f(x) = \begin{cases} x^2 & x < 0 \\ 2x & 0 \leq x < 2 \\ 4 & x \geq 2 \end{cases}$

f) $f(x) = \begin{cases} -(x+1)^2 & x < -1 \\ x^2 + 2 & -1 \leq x < 2 \\ -x^2 + 5 & x \geq 2 \end{cases}$

4. Identify the following functions as even, odd, or neither.

a) $f(x) = x^4 + 2$

b) $f(x) = x^2 + 2x$

c) $f(x) = x^5 + 4x^3 - x$

d) $f(x) = \sin(x)$

e) $f(x) = \cos(x)$

f) $f(x) = |x|$

5. Identify the center and radius of the given circles.

a) $(x-2)^2 + (y+5)^2 = 16$

b) $x^2 + y^2 + 4y + 4 = 25$

c) $x^4 - 6x + y^2 + 2x - 39 = 0$

d) $x^2 + 4x + y^2 = 0$

6. For each pair, P, Q, give the distance between P and Q, as well as the midpoint of the segment $\overline{PQ}$.

a) $P = (0,3)$, $Q = (1,-4)$.

b) $P = (2,5)$, $Q = (10,13)$.

c) $P = (1,4)$, $Q = (10,4)$.

d) $P = (0,0)$, $Q = (3,4)$.

Hint: All critical and inflection points are Integers.

MATH 135: Quiz 9

November 4, 2014

10

Name: Solutions Sec: ____________

Fill in the following table and use it to sketch the graph of the function $f(x)$ below.

$$f(x) = \frac{(x+2)(x-2)^2}{x} \qquad f'(x) = \frac{2(x-2)\overbrace{(x^2+x+2)}^{A}}{x^2} \qquad f''(x) = \frac{2(x+2)\overbrace{(x^2-2x+4)}^{B}}{x^3}$$

For each row in the table, list the interval(s) or point(s) where f has the given property. If none exist, write "none". The axes for the sketch are on the back of this page.

Roots ($f(x) = 0$)	$x = 2, \; x = -2$	(1)
Increasing	$(2, \infty)$	(1)
Decreasing	$(-\infty, 0) \cup (0, 2)$	
Concave Up	$(-\infty, -2) \cup (0, \infty)$	(1)
Concave Down	$(-2, 0)$	
Critical Points	$x = 2$ (Minimum)	(1)
Inflection Points	$x = -2$	
Horizontal Asymptotes	None	(1)
Vertical Asymptotes	$x = 0$	

A, B do not factor, so by the hint they are never 0.
By plugging in 0, we see that $A > 0$, $B > 0$ for all x

→ You can also plug in points to check this in each case (1)

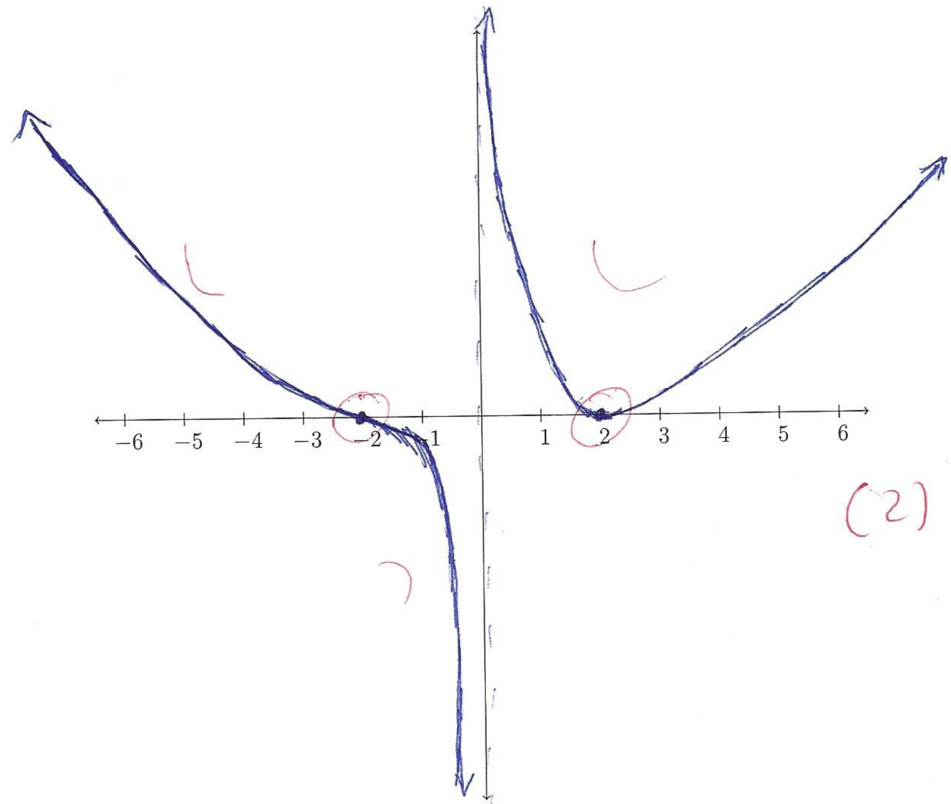
f: $\frac{(-)(-)^2}{(-)} = +$ | -2 | $\frac{(+)(-)^2}{(-)} = -$ | 0 | $+$ | 2 | $+$

or $\lim\limits_{x \to 0^-} \frac{(x+2)(x-2)^2}{x} = -\infty$

$\lim\limits_{x \to 0^+} \frac{(x+2)(x-2)^2}{x} = +\infty.$

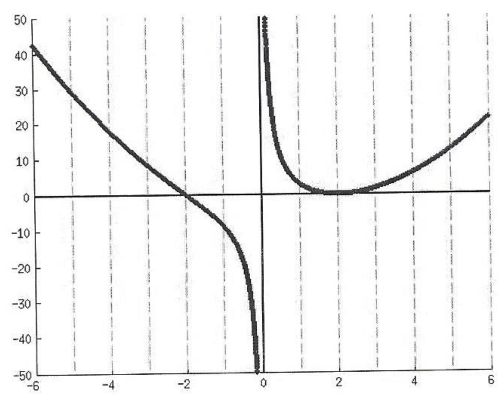
f': $-$ | 0 | $-$ | 2 | $+$ (1)

f'': $+$ | -2 | $-$ | 0 | $+$ | 1 (1)



Actual Graph:

Asymptote



Quiz 9 Detailed Write-up

$f(x) = \dfrac{(x+2)(x-2)^2}{x}$ $\quad f'(x) = \dfrac{2(x-2)(x^2+x+2)}{x^2}$ $\quad f''(x) = \dfrac{2(x+2)(x^2-2x+4)}{x^3}$

Hints : 1. Roots = X - intercepts

2. All critical points are integers

3. $x^2+x+2>0$ $\quad x^2-2x+4>0$ all x

and they don't factor.

- Roots / X - intercepts : Set $f(x)=0$ and solve.

$$0 = \frac{(x+2)(x-2)^2}{x}$$

$$0 = (x+2)(x-2)^2$$ ← Numerator must be zero.

$$\boxed{x=-2, \quad x=+2}$$

So these are the roots. ✓

- Vertical Asymptotes - Look where f is undefined.
  - The only place f is undefined is at $x=0$.
  - As $x \to 0$, the numerator goes to $(0-2)(0+2)^2 = 8$ and the denominator goes to zero.
  - Therefore, this is a vertical asymptote, and the only one since f is defined everywhere else.

$$\boxed{\text{Vertical Asymptote: } x = 0}$$

- <u>Horizontal Asymptotes</u> – $\lim_{x \to \infty} f(x)$

Method 1) The numerator of f goes like x^3, while the denominator is just x.

Therefore, there is no asymptote.

Method 2) Calculate the limit.

$$\lim_{x \to \infty} \frac{(x+2)(x-2)^2}{x} = \lim_{x \to \infty} \frac{x^3 - 2x^2 - 4x + 8}{x} \cdot \frac{(1/x^3)}{(1/x^3)}$$

$$= \lim_{x \to 0} \frac{1 - 2/x - 4/x^2 + 8/x^3}{1/x^2}$$

Taking the limit of each term "=" $\dfrac{1 - 0 - 0 + 0}{0} = \infty$ or undef.

Since this limit is undefined / ∞, there is no horizontal Asymptote.

$$\boxed{\text{Horizontal Asymptote: None}}$$

- Critical Points

  - 1^{st} order: $f'(x) = 0$ and solve

$$0 = \frac{2(x-2)(x^2+x+2)}{x^2}$$

$$0 = 2(x-2)(x^2+x+2)$$ — Numerator must be zero

$x-2=0$ or $x^2+x+2=0$

$x=2$ NO SOLUTIONS

By the hints given in class

$$\begin{cases} x^2+x+2 > 0 \quad \text{all } x \text{ ; or} \\ \text{All critical points are integers + } x^2+x+2 \text{ does not factor} \end{cases}$$

We get that x^2+x+2 is Never 0.

→ Quadratic Formula can also check that.

Therefore, the only 1^{st} order critical point is at $x=2$ (and the asymptote at $x=0$)

$$\boxed{\text{Critical Points} \quad x=2}$$

- 2nd Order : $f''(x) = 0$

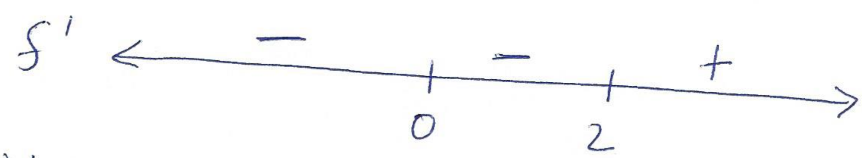
$$0 = \frac{2(x+2)(x^2-2x+4)}{x^3}$$

$$0 = 2(x+2)(x^2-2x+4)$$

$x = -2$ | NO SOLUTIONS by the Same argument for 1st order.

$$\boxed{\text{2nd Order (possible inflection point)} \quad x = -2}$$

- <u>Increasing/Decreasing</u> : Check f' in intervals
  - We need to mark the asymptote and critical points.



- You can either do this by positive/negative arguments, or plugging in points.

$$f'(-1) = \frac{2(-1-2)((-1)^2-1+2)}{(-1)^2} = \frac{2(-3)(1-1+2)}{1} = -12$$

$$f'(1) = \frac{2(1-2)(1^2+1+2)}{1^2} = -8$$

$$f'(3) = \frac{2(3-2)(3^2+3+2)}{3^2} = \frac{2 \cdot 1 \cdot 14}{9} = 28/9$$

and: $x = 2$ is a minimum

$$\boxed{\text{Increasing: } (2, \infty) \qquad \text{Decreasing: } (-\infty, 0) \cup (0, 2)}$$

- Concave Up / Down : Check f'' on intervals

  - Again mark the 2^{nd} order critical point(s) and asymptote.

f''

$(-\infty, -2)$	-2	$(-2, 0)$	0	$(0, \infty)$
$+$		$-$		$+$

$$f''(-3) = \frac{2(-3+2)((-3)^2-2(-3)+4)}{(-3)^3} = \frac{2\cdot(-1)\cdot(9+6+4)}{-27} = \frac{38}{27}$$

$$f''(-1) = \frac{2(-1+2)((-1)^2-2(-1)+4)}{(-1)^3} = \frac{2(1)(1+2+4)}{-1} = -14$$

$$f''(1) = \frac{2(1+2)(1^2-2+4)}{1^3} = \frac{2\cdot3\cdot3}{1} = 18$$

$$\boxed{\begin{array}{l}\text{Concave up : } (-\infty,-2) \cup (0,\infty) \\ \text{Concave down : } (-2,0)\end{array}}$$

and

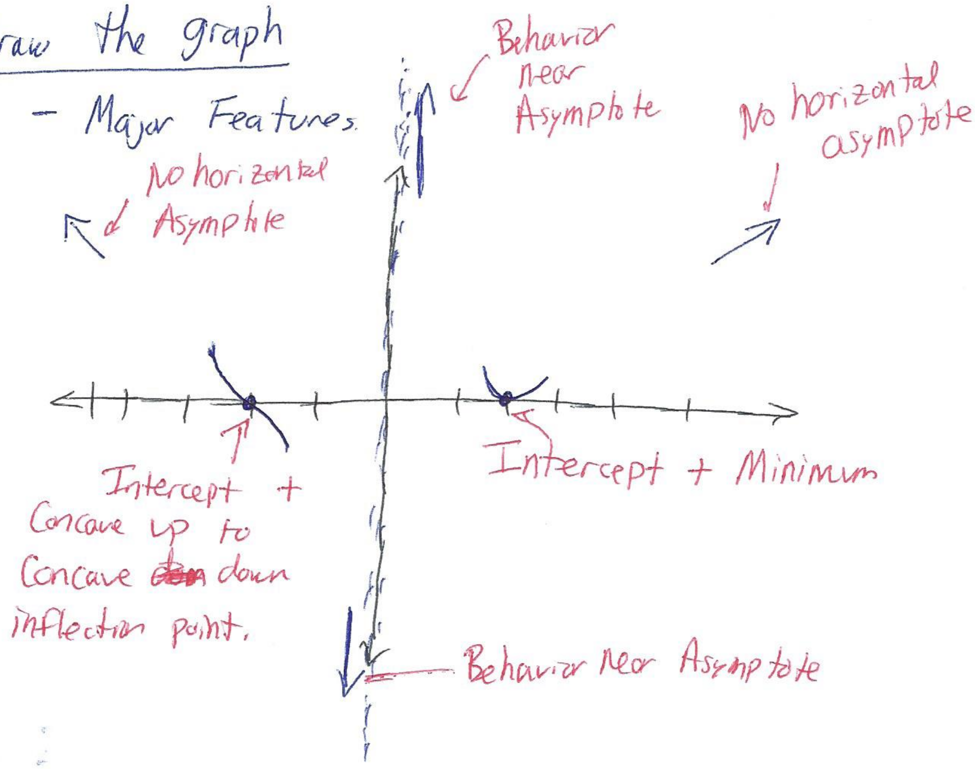
$$\boxed{x = -2 \text{ is an inflection point}}$$

- Behavior near the asymptote

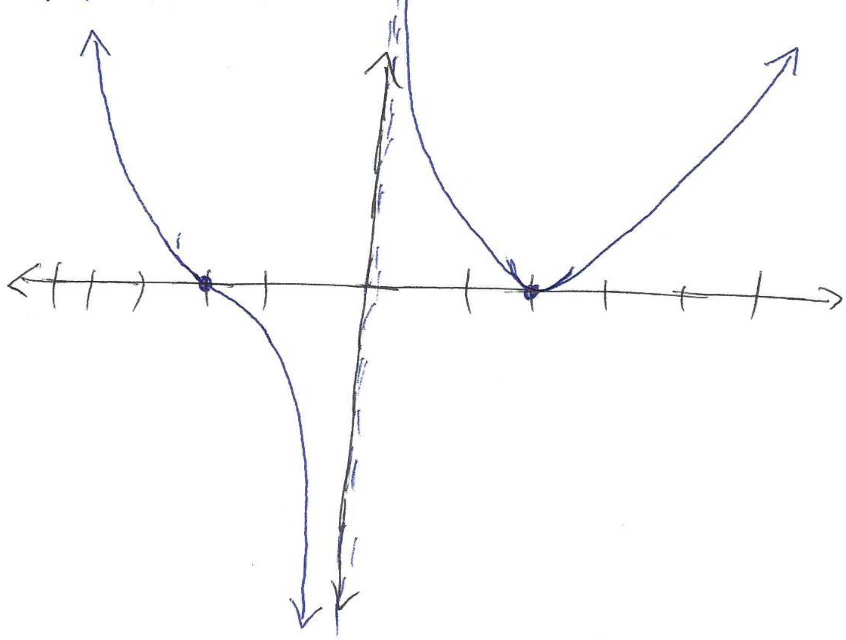
  - As $x \to 0^-$, $f'(x) < 0$, so the graph must go down to $-\infty$ on the left side

  - As $x \to 0^+$, $f'(x) < 0$, so the graph must come down from $+\infty$ on the right.

- Draw the graph
  - Major Features.



  - Fill in the curve.



MATH 135: Final Exam Concept Review

Matt Charnley

December 8, 2014

Chapter 1

(a) Section 1.2

  (i) Absolute value equalities and inequalities.

  (ii) Distance and midpoint. Circles.

  (iii) Trigonometric equations.

(b) Section 1.3

  (i) Slope and equation of lines.

  (ii) Parallel and perpendicular. Intercepts.

(c) Section 1.4

  (i) Function notation. Composition of functions.

  (ii) Basic graphs. Intercepts and Symmetry.

  (iii) Polynomial and rational functions.

Chapter 2

(a) Section 2.1

  (i) Definition and notation of limit.

  (ii) One-sided limits.

  (iii) Reading limits from graph.

  (iv) Infinite limits.

(b) Section 2.2

  (i) Algebraic rules for limits.

  (ii) Factoring and canceling.

  (iii) Piecewise defined functions.

  (iv) Special Trig limits.

$$\lim_{h\to 0}\frac{\sin h}{h} = 1 \qquad \lim_{h\to 0}\frac{\cos h - 1}{h} = 0$$

  (v) Trig Limits.

  (vi) Squeeze Theorem

(c) Section 2.3

  (i) Definition of continuity

  (ii) Continuity of polynomials and rational functions.

(iii) Intermediate Value Theorem.

(d) Section 2.4

  (i) Exponential Function
  (ii) Logarithm function.
  (iii) Solving logarithmic equations.
  (iv) Compounding interest / Biological growth.

Chapter 3

(a) Section 3.1

  (i) Lines tangent to graph.
  (ii) Definition of derivative.
  (iii) Graphs of f and f'.
  (iv) Existence of derivatives.

(b) Section 3.2

  (i) Algebraic rules for derivatives.
  (ii) Product and Quotient rules.
  (iii) Higher order derivatives.

(c) Section 3.3

  (i) Trig derivatives.
  (ii) Derivatives of exponential and logs.

(d) Section 3.4

  (i) Average, instantaneous, relative rates of change.
  (ii) Rectilinear Motion
  (iii) Falling Body Problems

(e) Section 3.5

  (i) Chain Rule
  (ii) "Extended Derivative" Rules

(f) Section 3.6

  (i) Implicit Differentiation
  (ii) Logarithmic differentiation
  (iii) Finding tangent lines this way.

(g) Section 3.7

  (i) Related Rates
  (ii) All the different types of problems.

(h) Section 3.8

  (i) Tangent Line Approximations
  (ii) Differentials
  (iii) Error propagation

Chapter 4

(a) Section 4.1

  (i) Absolute and relative maxima and minima.
  (ii) Critical points/numbers
  (iii) Max/min or optimization on a closed interval.

(b) Section 4.2

  (i) Mean value theorem.
  (ii) How to use it.

(c) Section 4.3

  (i) Graph Sketching - Increasing/Decreasing
  (ii) Concave up/down.
  (iii) First and second derivative test.
  (iv) Inflection points.
  (v) Using this to draw graphs.

(d) Section 4.4

  (i) Limits at infinity - Horizontal Asmyptotes
  (ii) Infinite limits - Vertical Asmyptotes
  (iii) Vertical tangents.

(e) Section 4.5

  (i) L'Hôpital's Rule
  (ii) When you can and can't apply it.
  (iii) Manipulating to get things in the right form.
  (iv) Limits involving e and ln.

(f) Section 4.6

  (i) Optimization for physical problems.
  (ii) Draw a picture.
  (iii) Derive and equation.
  (iv) Maximize or minimize.

(g) Section 4.7

  (i) More optimization. Same things, different terms.
  (ii) Economics (revenue, profit, costs, marginal...)
  (iii) Physiology.

Chapter 5

(a) Section 5.1

   (i) Antiderivatives
   (ii) Indefinite integrals.
   (iii) Rules for antiderivatives.
   (iv) Applications.

(b) Section 5.2

   (i) Area as the limit of a sum. Add up little rectangles.
   (ii) Sigma notation.

(c) Section 5.3

   (i) Riemann Sums
   (ii) Definite integral as the limit of Riemann Sums.
   (iii) Integral at a point and changing order of limits.
   (iv) Area and distance as integrals.

(d) Section 5.4

   (i) FTC - version 1. Integral is the difference of anti-derivative.
   (ii) FTC - version 2. Derivative of the integral gives the function back.

(e) Section 5.5

   (i) u substitution.
   (ii) Make the inside u. Don't forget to find du.
   (iii) Change limits of integration for a definite integral.

A.2 Spring 2015 - Math 251

This section contains the following documents from my teaching assistant assignment for Math 251 in Spring 2015:

1. My recitation syllabus handed out during the first week of class
2. Solutions to a sample quiz given during recitation
3. An example document on Clairaut's theorem that was distributed to the class to illustrate that the conditions of this theorem are necessary

Math 251: Recitation Syllabus

Teaching Assistant: Matt Charnley
Email: charnley@math.rutgers.edu
Website: http://math.rutgers.edu/~mpc163/Courses/SP15_MATH251.html
Office: Hill 606, Busch Campus
Office Hours:

- Mondays, 12:00 noon – 1:30 pm
- Wednesdays, 3:15 pm – 4:45 pm
- By appointment

Class Times: Section 22: Thursdays, 4:30-5:50, SC-205
Section 23: Thursdays, 6:10-7:30, SC-120
Section 24: Thursdays, 7:40-9:00, SC-120

Goal: The goal of these recitation sessions is to give extra help with solving problems from the textbook. An important part of Calculus 3 is learning how to attack the large variety of problems and feel comfortable with the concepts involved. With recitation on Thursdays, and homework due on Mondays, the discussion in recitation will cover the material from Monday and Wednesday's lectures of the same week. I would recommend looking over the material from Wednesday's lecture before recitation so that you can have questions answered about those topics. Recitation will consist of a mixture of topic reviews and problem solving at the board, depending on the topics and interest of the class.

Quizzes: Quizzes will be given approximately every other week, but this may vary slightly depending on the test schedule and other factors. Quizzes will be about the material discussed in that day's recitation. Make-up quizzes will not be offered. As of now, two quizzes will be dropped over the course of the semester, so missing a week or two will not affect your final grade for the class. Quiz solutions will be posted both on Sakai and on my website.

Maple: Over the course of the semester, there will be 5 Maple Labs that will need to be turned in, in addition to Lab 0, which you have right now. They will be assigned over the course of the semester, and you will have two weeks to complete each one once they are assigned. Outside of the first recitation in the computer lab, all of the work on the Maple Labs will take place outside of class. I will be happy to answer questions about them via email or in office hours, but will not dedicate recitation time to it.

Course Details: Information for the course will be primarily posted on Sakai. There will also be information on my website above, but it will all be duplicated from Sakai. For more information about the course, please see the syllabus or other information posted on Sakai

MATH 251: Quiz 8

April 30, 2015

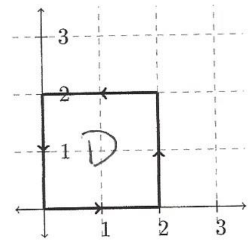
Name: Solutions Sec: ________

/5

1. Use Green's Theorem to calculate the integral $\oint_C \vec{F} \cdot d\vec{s}$ for the vector field

$$\vec{F} = \langle 2xy + x^4, 3xy^2 - \sin(y) \rangle$$

and the curve



By Green's Thm. $\displaystyle\oint_C \vec{F} \cdot d\vec{s} = \iint_D \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} \, dA$ (2)

$\displaystyle\frac{\partial F_2}{\partial x} = 3y^2 \qquad \frac{\partial F_1}{\partial y} = 2x$ (1)

So $\displaystyle\oint_C \vec{F} \cdot d\vec{s} = \int_0^2 \int_0^2 3y^2 - 2x \, dx \, dy$ (1)

$\displaystyle = \int_0^2 3xy^2 - x^2 \Big|_0^2 dy$

$\displaystyle = \int_0^2 6y^2 - 4 \, dy = 2y^3 - 4y \Big|_0^2$

$= 16 - 8 = \boxed{8}$ (1)

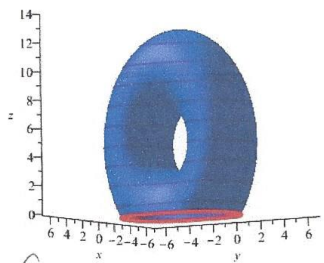
2. Use Stokes' Theorem to evaluate the integral

$$\iint_{\mathcal{S}} \operatorname{curl}(\vec{F}) \cdot d\vec{S}$$

for the surface $\mathcal{S}$ with outward normal vector and vector field $\vec{F}$ below, where the boundary of $\mathcal{S}$ is the ellipse $4x^2 + y^2 = 16$ in the xy-plane. This boundary can be parametrized as $c(t) = \langle 2\cos(t), 4\sin(t), 0\rangle$. $0 \le t \le 2\pi$

/5

$$\vec{F} = \langle 3x + 4zx^2, x + y + z, x^2 + y^2 + z^2\rangle$$



By Stokes' thm: $\displaystyle\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s}$ (1)

$c(t) = \langle 2\cos(t), 4\sin(t), 0\rangle$

$c'(t) = \langle -2\sin(t), 4\cos(t), 0\rangle$ (1)

$F(c(t)) = \langle 6\cos t + 0, 2\cos t + 4\sin t + 0, 4\cos^2 t + 16\sin^2 t + 0\rangle$

$F(c(t)) \cdot c'(t) = -12\cos(t)\sin t + 8\cos^2 t + 16\sin t\cos t + 0$

$= 4\cos t \sin t + 8\cos^2 t$

$= 4\cos t\sin t + 4 + 4\cos 2t$ (1)

So $\displaystyle\iint_S \text{curl}(\vec{F}) \cdot d\vec{S} = \oint_{\partial S} \vec{F} \cdot d\vec{s} = \int_0^{2\pi} \vec{F}(c(t)) \cdot c'(t)\, dt = \int_0^{2\pi} 4\cos t \sin t + 4 + 4\cos 2t\, dt$ (1)

$= 2\sin^2 t + 4t + 2\sin 2t \Big|_0^{2\pi} = 8\pi$ (1)

Clairaut's Theorem

Matt Charnley

February 20, 2015

In this document, we describe a function that does not satisfy the hypotheses of Clairaut's theorem, and therefore, the mixed second partial derivatives are not equal.

Let

$$f(x, y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

with $f(0,0) = 0$. We compute the first partial derivatives

$$f_x(x,y) = \begin{cases} \frac{y(x^2-y^2)+2x^2y}{(x^2+y^2)} - \frac{2x^2y(x^2-y^2)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and

$$f_y(x,y) = \begin{cases} \frac{x(x^2-y^2)+2xy^2}{(x^2+y^2)} - \frac{2xy^2(x^2-y^2)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

where we needed to look at the actual function f and take limits to get the value of these derivatives at $(0,0)$. To compute the second derivatives at $(0,0)$ we follow this same approach.

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h \to 0} \frac{(-h+0) - 0}{h} = -1$$

since, plugging $x = 0$ into the equation for f_x gives

$$f_x(0,y) = \frac{y(0-y^2)+0}{0+y^2} - 0 = -y.$$

Doing the same thing for f_{yx} we see that

$$f_{yx}(0,0) = \lim_{h\to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h\to 0} \frac{(h+0)-0}{h} = 1.$$

Thus, we have that $f_{xy} \neq f_{yx}$ at the point $(0,0)$. This is because neither of these derivatives are continuous in a disk containing $(0,0)$.

A.3 Fall 2015 - Math 251

This section contains the following documents from my teaching assistant assignment for Math 251 in Fall 2015:

1. My recitation syllabus handed out the first week of class
2. My written-up solution to part of the midterm that was given during my recitation session
3. Images of the lecture notes for the first section of extra video lectures that I recorded to make up for missed material in class

Math 251: Recitation Syllabus

Teaching Assistant: Matt Charnley
Email: charnley@math.rutgers.edu
Website: http://math.rutgers.edu/~mpc163/Courses/FA15_MATH251.html
Office: Hill 606, Busch Campus
Office Hours:

- Tuesdays, 3:15 pm – 4:45 pm
- Wednesdays, 12:00 noon – 1:30 pm
- By appointment

Class Times: Section H1: Thursdays, 3:20-4:40, BE-251
Section H2: Thursdays, 5:00-6:20, BE-013
Section H3: Thursdays, 6:40-8:00, BE-013

Goal: The goal of these recitation sessions is to give extra help with solving problems from the textbook. An important part of Calculus 3 is learning how to attack the large variety of problems and feel comfortable with the concepts involved. The discussion in recitation will cover the material from Monday and Wednesday's lectures of the same week. I would recommend looking over the material from Wednesday's lecture before recitation so that you can have questions answered about those topics. Recitation will consist of a mixture of topic reviews and problem solving at the board, depending on the topics and interest of the class.

Quizzes: Quizzes will be given approximately every other week, but this may vary slightly depending on the test schedule and other factors. Quizzes will cover all material up until the Monday before recitation. Make-up quizzes will not be offered. Two quizzes will be dropped over the course of the semester, so missing a week or two will not affect your final grade for the class. Quiz solutions will be posted on my website.

Maple: Over the course of the semester, there will be 5 Maple Labs that will need to be turned in, in addition to Lab 0, which you have right now. They will be assigned over the course of the semester, and you will have two weeks to complete each one once they are assigned. Outside of the first recitation in the computer lab, all of the work on the Maple Labs will take place outside of class. I will be happy to answer questions about them via email or in office hours, but will not dedicate recitation time to it.

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MATH 251: In-Class Midterm

December 3, 2015

Name: Solutions Sec: ____________

1. Let $\mathcal{D}$ be the region between the curves $y = 5 - x^2$ and $y = x^2 - 3$. Sketch the region and compute the integral of $f(x, y) = x^2$ over this region.

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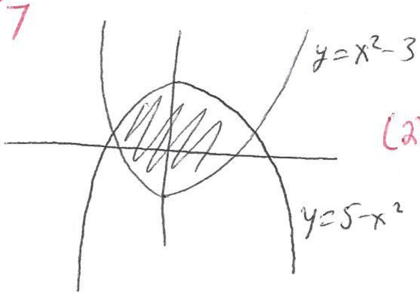


(2)

$$\int_{-2}^{2} \int_{x^2-3}^{5-x^2} x^2 \, dy \, dx$$

$$= \int_{-2}^{2} x^2 y \Big|_{x^2-3}^{5-x^2} dx$$

(2)

Intersections:

$5 - x^2 = x^2 - 3$

$2x^2 = 8$

$x^2 = 4$

$x = \pm 2$

$$= \int_{-2}^{2} 5x^2 - x^4 - x^4 + 3x^2 \, dx$$

$$= \int_{-2}^{2} 8x^2 - 2x^4$$

$$= \frac{8}{3}x^3 - \frac{2}{5}x^5 \Big|_{-2}^{2}$$

(2)

$$= \frac{8}{3}(2)^3 - \frac{2}{5}(2)^5 + \frac{8}{3}(+2)^3 \mp \frac{2}{5}(+2)^5$$

(1)

$$= \frac{16 \cdot 8}{3} - \frac{4 \cdot 32}{5} = 128\left(\frac{1}{3} - \frac{1}{5}\right) = \boxed{\frac{256}{15}}$$

2. Compute the volume of the region $\mathcal{R}$ sitting above the triangle bounded by $x = 0$, $y = 0$ and $y = 1 - x$ in the xy-plane, and between the planes, $x + y + z = 5$ and $2x + y + 3z = 6$.

/6

$z = 5 - x - y \qquad z = \frac{1}{3}(6 - 2x - y)$

$$\int_0^1 \int_0^{1-x} \int_{\frac{1}{3}(6-2x-y)}^{5-x-y} 1 \, dz \, dy \, dx \quad (2)$$

$$= \int_0^1 \int_0^{1-x} 5 - x - y - \frac{1}{3}(6 - 2x - y) \, dy \, dx$$

$$= \int_0^1 \int_0^{1-x} 3 - \frac{1}{3}x - \frac{2}{3}y \, dy \, dx \quad (2)$$

$$= \int_0^1 3y - \frac{1}{3}xy - \frac{1}{3}y^2 \Big|_0^{1-x} dx$$

$$= \int_0^1 3(1-x) - \frac{1}{3}x + \frac{1}{3}x^2 - \frac{1}{3}(1-x)^2 \, dx \quad (1)$$

$$= -\frac{3}{2}(1-x)^2 - \frac{1}{6}x^2 + \frac{1}{9}x^3 + \frac{1}{9}(1-x)^3 \Big|_0^1$$

$$= \frac{1}{9} - \frac{1}{6} + \frac{3}{2} - \frac{1}{9} = 8/6 = \boxed{4/3} \quad (1)$$

$\rho = 4$

3. Find the integral of $f(x, y, z) = x + z$ over the region inside the hemisphere of radius 4 where $y \geq 0$, and above the plane $z = 2$.

/7

$0 \le \theta \le \pi$

$z = 2 \rightarrow \rho\cos\varphi = 2$

$\rho = 2/\cos\varphi$

$4 = \rho = 2/\cos\varphi$

$\cos\varphi = 1/2$ (2)

$\varphi = \pi/3$

$$\rightarrow \int_0^{\pi/3}\int_0^{\pi}\int_{2/\cos\varphi}^{4} (\rho\cos\theta\sin\varphi + \rho\cos\varphi)\rho^2\sin\varphi \, d\rho\, d\theta\, d\varphi$$ (2)

$$= \int_0^{\pi/3}\int_0^{\pi}\int_{2/\cos\varphi}^{4} \rho^3\cos\theta\sin^2\varphi + \rho^3\cos\varphi\sin\varphi \, d\rho\, d\theta\, d\varphi$$

$$= \int_0^{\pi/3}\int_0^{\pi} \frac{\rho^4}{4}\left(\cos\theta\sin^2\varphi + \cos\varphi\sin\varphi\right)\Big|_{2/\cos\varphi}^{4} d\theta\, d\varphi$$

$$= \int_0^{\pi/3}\int_0^{\pi}\left(64 - \frac{4}{\cos^4\varphi}\right)\left(\cos\theta\sin^2\varphi + \cos\varphi\sin\varphi\right) d\theta\, d\varphi$$ (1)

$$= \int_0^{\pi/3}\left(64 - \frac{4}{\cos^4\varphi}\right)\left(\sin\theta\sin^2\varphi + \theta\cos\varphi\sin\varphi\right)\Big|_0^{\pi} d\varphi$$ (1)

$$= \pi\int_0^{\pi/3} 64\cos\varphi\sin\varphi - 4\frac{\sin\varphi}{\cos^3\varphi}\, d\varphi$$

$u = \cos\varphi$

$du = -\sin\varphi\, d\varphi$

$$= \pi\left[-32\cos^2\varphi - 2\cos^{-2}\varphi\right]_0^{\pi/3} = \pi\left[-\frac{32}{4} - 2\cdot 4 + 32 + 2\right]$$

$$= \boxed{18\pi}$$ (1)

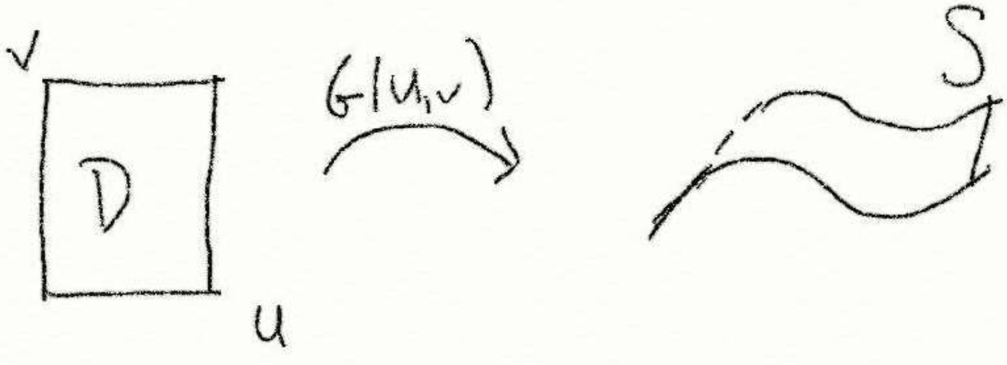
Section 16.4: Parameterized Surfaces and Surface Integrals

16.2: Parametrized curve

$$C(t) = \langle x(t), y(t), z(t) \rangle$$

<u>Parametrized Surface</u> $S = G(u, v)$

$$G(u,v) = \langle x(u,v), y(u,v), z(u,v) \rangle$$



<u>Ex</u> (r, θ, z) fix $r = R$

$x = r\cos\theta$

$y = r\sin\theta$

$z = z$

$2\cos u \quad 2\sin u \quad v$

$$G(u,v) = \langle R\cos u,\ R\sin u,\ v\rangle$$

$$D = [0, 2\pi] \times \mathbb{R} \quad [1,5]$$

(ρ, θ, φ) $\quad \rho = R$

$x = \rho\cos\theta\sin\varphi$

$y = \rho\sin\theta\sin\varphi$

$z = \rho\cos\varphi$

$$G(u,v) = \langle R\cos u\sin v,\ R\sin u\sin v,\ R\cos v\rangle$$

$$D = [0, 2\pi] \times [0, \pi]$$

<u>Cone</u> $\quad z^2 = x^2 + y^2 = r^2 \rightarrow z = \pm r$

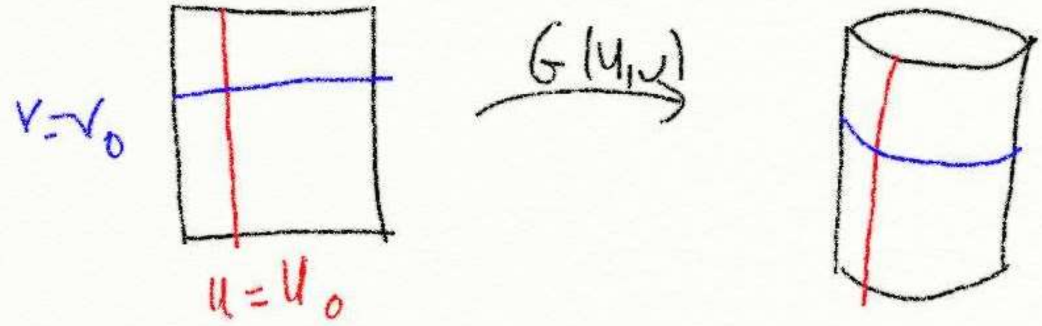
$x = r\cos\theta$

$y = r\sin\theta$

$$G(u,v) = \langle \underline{u\cos v},\ \underline{u\sin v},\ \underline{u}\rangle$$

Graph of $z = f(x,y)$

$$G(u,v) = \langle u, v, f(u,v) \rangle$$



curve $G(u_0, v) = \langle x(u_0, v), y(u_0, v), z(u_0, v) \rangle$

Tangent Vector $\underline{T_v(u_0, v)} = \frac{\partial G}{\partial v}(u_0, v)$

$$= \left\langle \frac{\partial x}{\partial v}(u_0, v), \frac{\partial y}{\partial v}(u_0, v), \frac{\partial z}{\partial v}(u_0, v) \right\rangle$$

$\underline{T_u}(u,v) = \frac{\partial G}{\partial u}(u,v) = \left\langle \frac{\partial x}{\partial u}(u,v), \frac{\partial y}{\partial u}(u,v), \frac{\partial z}{\partial u}(u,v) \right\rangle$

$\vec{n}(u,v) = \vec{T}_u(u,v) \times \vec{T}_v(u,v)$

Tangent Plane: $\vec{n} \cdot \langle x - x_0, y - y_0, z - z_0 \rangle = 0$

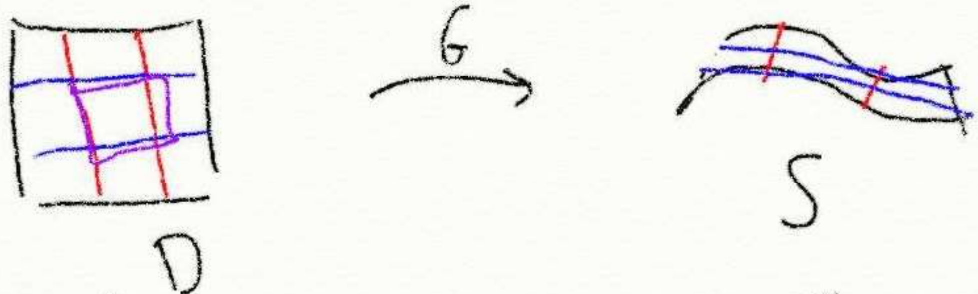
Ex $G(u,v) = \langle u, uv, v^2 + 1 \rangle$

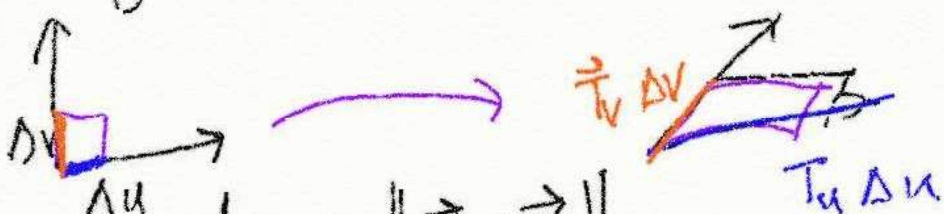
$T_u = \langle 1, v, 0 \rangle$

$$T_v = \langle 0, u, 2v \rangle$$

$$\vec{n} = \langle 2v^2, -2v, u \rangle = T_u \times T_v$$

Surface Area





$$\text{Area} = \|\vec{v}_1 \times \vec{v}_2\|$$

$$\Delta u \longrightarrow \vec{T}_u \Delta u$$

$$\Delta v \longrightarrow T_v \Delta v$$

$$\text{Area} = \|\vec{T}_u \Delta u \times \vec{T}_v \Delta v\|$$

$$= \|\vec{n}\| \, \Delta u \, \Delta v$$

Surface S, Domain D, $G(u,v)$

$$\boxed{\iint_S f\,dS = \iint_D f(G(u,v))\,\|\vec{n}(u,v)\|\,du\,dv}$$

$$\text{Area}(S) = \iint_D \|\vec{n}(u,v)\|\,du\,dv$$

$$G(u,v) = \langle x(u,v),\, y(u,v),\, 0\rangle$$

$$\vec{T}_u = \left\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}, 0\right\rangle$$

$$T_v = \left\langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}, 0\right\rangle$$

$$\vec{n} = \left\langle 0, 0, \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right\rangle$$

$$\|\vec{n}\| = \left| \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial x}{\partial v}\frac{\partial y}{\partial u}\right| = |\text{Jac}(G)|$$

Ex 1 Find the surface area of the upper hemisphere of radius R, and integrate z over this surface

$$G(u,v) = \langle R\cos u \sin v,\ R\sin u \sin v,\ R\cos v\rangle$$

$$D = [0, 2\pi] \times [0, \pi/2]$$

$$\vec{T}_u = \langle -R \sin u \sin v, \ R \cos u \sin v, \ 0 \rangle$$

$$T_v = \langle R \cos u \cos v, \ R \sin u \cos v, \ -R \sin v \rangle$$

$$\vec{n} = \langle -R^2 \cos u \sin^2 v, \ -R^2 \sin u \sin^2 v, \ -R^2 \underline{\sin^2 u} \sin v \cos v$$
$$- R^2 \underline{\cos^2 u} \sin v \cos$$

$$= -R^2 \langle \cos u \sin^2 v, \ \sin u \sin^2 v, \ \cos v \sin v \rangle$$

$$= -R^2 \sin v \ \underline{\langle \cos u \sin v, \ \sin u \sin v, \ \cos v \rangle}$$

$$\|\vec{n}\| = R^2 \sin v \qquad 1$$

$$\text{Area} = \int_0^{2\pi} \int_0^{\pi/2} R^2 \sin v \, dv \, du$$

$$= 2\pi \cdot R^2 \cdot -\cos v \Big|_0^{\pi/2} = \underline{2\pi R^2}$$

$$\iint_S z \, dS = \int_0^{\pi} \int_0^{\pi/2} R \cos v \cdot R^2 \sin v \, dv \, du$$

$$= 2\pi R^3 \cdot \frac{\sin^2 v}{2}\Big|_0^{\pi/2}$$

$$= \boxed{\pi R^3}$$

Given the surface

$$G(u,v) = \langle u, u+v, v^2 \rangle \quad [0,1]\times[0,1]$$

$\iint_S y-x \, dS$. Find the tangent plane at $u=1, v=1$

$$T_u = \langle 1, 1, 0 \rangle$$

$$T_v = \langle 0, 1, 2v \rangle$$

$$\vec{n} = \langle 2v, -2v, 1 \rangle$$

$$\|\vec{n}\| = \sqrt{4v^2 + 4v^2 + 1} = \sqrt{8v^2+1}$$

$$\iint_S y-x \, dS = \int_0^1 \int_0^1 v+u-u \sqrt{8v^2+1} \, du \, dv$$

$$= \int_0^1 \int_0^1 v \sqrt{8v^2+1} \, du \, dv$$

$w = 8v^2+1$

$dw = 16v\,dv$

$$= \int_0^1 v\sqrt{8v^2+1}\,dv$$

$$= \frac{1}{16}\int_1^9 \sqrt{w}\,dw$$

$$= \frac{1}{16} w^{3/2} \cdot \frac{2}{3} \Big|_1^9$$

$$= \frac{1}{24}\left[27-1\right] = \frac{26}{24} = \boxed{\frac{13}{12}}$$

$\vec{n} = \langle 2v, -2v, 1 \rangle$

at $u=1, v=1$ $\quad \vec{n} = \langle 2, -2, 1 \rangle$

$G(1,1) = \langle 1, 2, 1 \rangle$

$$2(x-1) - 2(y-2) + (z-1) = 0$$

A.4 Spring 2016 - Math 575

This section contains the following documents from my teaching assistant assignment for Math 575:

1. A sample homework solution done in MATLAB
2. A sample homework solution with more conceptual type problems
3. A sample piece of Python code from another homework set
4. A document outlining some properties of variational formulations, which I noticed the students were having trouble with

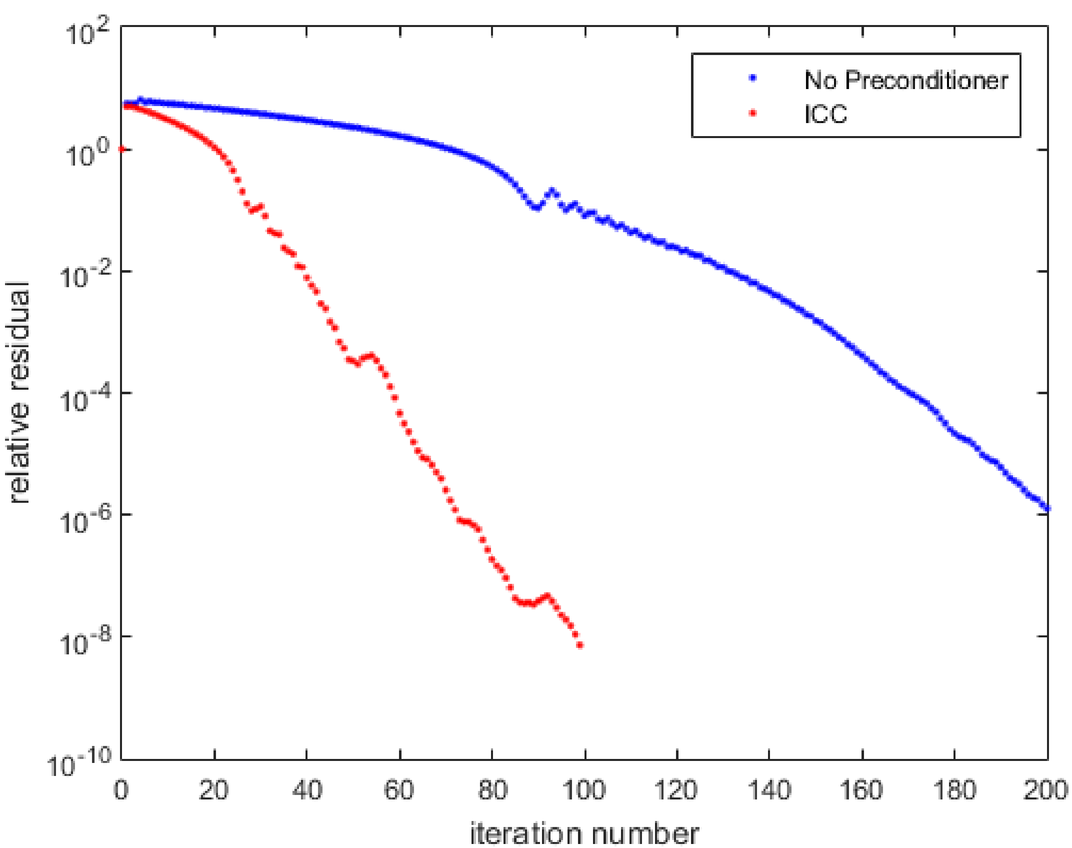
Math575HW2.m

Matt Charnley

Computes the solution to a given finite difference scheme. Builds the appropriate matrix and solves the linear system on a square.

```
f = @(x,y) 1;
g = @(x,y) 0;

min_coord = 0;
max_coord = 1;

N = 128;
x_val = linspace(min_coord, max_coord, N+1);
y_val = x_val;

h = (max_coord - min_coord)/N;

% The order for the coordinates will start from the bottom left, move
% over in x, then move up to the first y-coordinate, back on the far left
% in x, repeating until we get to the top right.

coord_x = zeros(1, (N+1)^2);
coord_y = zeros(1, (N+1)^2);

for i=1:(N+1)^2
   coord_x(i) = x_val(mod(i-1, N+1)+1);
   coord_y(i) = y_val(floor((i-1)/(N+1))+1);
end

A1 = zeros((N+1)^2, (N+1)^2);
b1 = zeros(1, (N+1)^2);

for i=1:(N+1)^2
   if (i <= N+1) || (i > N*(N+1)) || (mod(i, N+1) == 0) || (mod(i, N+1) == 1)
       A1(i,i) = 1;
       b1(i) = g(coord_x(i), coord_y(i));
   else
       A1(i,i) = 4 + h^2;
       A1(i,i-1) = -1;
       A1(i,i+1) = -1;
       A1(i, i+N+1) = -1;
       A1(i, i-N-1) = -1;
       b1(i) = f(coord_x(i), coord_y(i))*h^(2);
   end
end

% Try to cut out the unnecessary rows from the matrix. (i.e., the boundary
% terms
indSet = [];
for i = 1:(N+1)^2
    if A1(i,i) == 1
        indSet = [indSet, i];
```

```
        for j = 1:(N+1)^2
            if A1(j,i) == -1
                b1(j) = b1(j) + g(coord_x(i), coord_y(i));
            end
        end
    end
end

leftover = setdiff(1:(N+1)^2, indSet);

A_new = A1(leftover, leftover);
b_new = b1(leftover);

A2 = sparse(A_new);
[x0m, fl0m, rr0m, it0m, rv0m] = pcg(A2, b_new', 1e-8, 200);

Lm = ichol(A2);
[x1m, fl1m, rr1m, it1m, rv1m] = pcg(A2,b_new',1e-8, 200, Lm, Lm');

figure;
semilogy(0:it0m,rv0m/norm(b1),'b.');
hold on;
semilogy(0:it1m,rv1m/norm(b1),'r.');
legend('No Preconditioner','ICC');
xlabel('iteration number');
ylabel('relative residual');
hold off;
```



Published with MATLAB® R2015a

Math 575: HW 5

Matt Charnley

April 5, 2016

Problem 1. For this problem, we let ϕ_i be the basis functions for our given triangle with vertices (x_1, y_1), (x_2, y_2), and (x_3, y_3) and $\hat{\phi}_i$ be those for the reference triangle with vertices $(0, 0)$, $(1, 0)$ and $(0, 1)$. Also, (x, y) will be coordinates in the original triangle, and $(\hat{x}, \hat{y})$ will be coordinates in the reference triangle. We will first establish some relationships between these two sets of functions and coordinates.

Define the matrix

$$A = \begin{pmatrix} x_2 - x_1 & x_3 - x_1 \\ y_2 - y_1 & y_3 - y_1 \end{pmatrix} \qquad A^{-1} = \frac{1}{\det(A)} \begin{pmatrix} A_{22} & -A_{12} \\ -A_{21} & A_{11} \end{pmatrix}$$

Then, we know that

$$\begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} + \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \qquad \begin{pmatrix} \hat{x} \\ \hat{y} \end{pmatrix} = A^{-1} \begin{pmatrix} x - x_1 \\ y - y_i \end{pmatrix}$$

We can also see that (by shifting to the other coordinates) that

$$\begin{pmatrix} x - x_i \\ y - y_i \end{pmatrix} = A \begin{pmatrix} \hat{x} - \hat{x}_i \\ \hat{y} - \hat{y}_i \end{pmatrix}$$

or, by expanding this out, we see that

$$x - x_i = A_{11}(\hat{x} - \hat{x}_i) + A_{12}(\hat{y} - \hat{y}_i) \qquad y - y_i = A_{21}(\hat{x} - \hat{x}_i) + A_{22}(\hat{y} - \hat{y}_i)$$

Similarly, we have that

$$\nabla \phi_i = (A^{-1})^T \nabla \hat{\phi}_i$$

or

$$\partial_x \phi_i = (A^{-1})_{11} \partial_x \hat{\phi}_i + (A^{-1})_{21} \partial_y \hat{\phi}_i \qquad \partial_y \phi_i = (A^{-1})_{12} \partial_x \hat{\phi}_i + (A^{-1})_{22} \partial_y \hat{\phi}_i$$

or, using how A^{-1} is defined

$$\partial_x \phi_i = \frac{1}{\det(A)} [A_{22} \partial_x \hat{\phi}_i - A_{21} \partial_y \hat{\phi}_i] \qquad \partial_y \phi_i = \frac{1}{\det(A)} [-A_{12} \partial_x \hat{\phi}_i + A_{11} \partial_y \hat{\phi}_i]$$

The important thing to note is that all of these relations are linear. By carrying out the computations, it can be seen from these relations that if the set of formulas are satisfied for the reference triangle, then they are satisfied for the normal triangle. We can see that

$$
\begin{aligned}
\sum_{i=1}^{3} \phi_i(x,y) &= \sum_{i=1}^{3} \hat{\phi}_i(x,y) = 1 \\
\sum_{i=1}^{3} (x_i - x)\phi_i(x,y) &= \sum_{i=1}^{3} [A_{11}(\hat{x} - \hat{x}_i) + A_{12}(\hat{y} - \hat{y}_i)]\hat{\phi}_i(x,y) = A_{11}\sum_{i=1}^{3}(\hat{x} - \hat{x}_i)\hat{\phi}_i(x,y) + A_{12}\sum_{i=1}^{3}(\hat{y} - \hat{y}_i)\hat{\phi}_i(x,y) = 0 + 0 \\
\sum_{i=1}^{3} (y_i - y)\phi_i(x,y) &= \sum_{i=1}^{3} [A_{21}(\hat{x} - \hat{x}_i) + A_{22}(\hat{y} - \hat{y}_i)]\hat{\phi}_i(x,y) = A_{21}\sum_{i=1}^{3}(\hat{x} - \hat{x}_i)\hat{\phi}_i(x,y) + A_{22}\sum_{i=1}^{3}(\hat{y} - \hat{y}_i)\hat{\phi}_i(x,y) = 0 + 0 \\
\sum_{i=1}^{3} \partial_x \phi_i(x,y) &= \sum_{i=1}^{3} \frac{1}{\det(A)}[A_{22}\partial_x\hat{\phi}_i - A_{21}\partial_y\hat{\phi}_i] = \frac{A_{22}}{\det(A)}\sum_{i=1}^{3}\partial_x\hat{\phi}_i - \frac{A_{21}}{\det(A)}\sum_{i=1}^{3}\partial_y\hat{\phi}_i = 0 - 0 = 0 \\
\sum_{i=1}^{3} \partial_y \phi_i(x,y) &= \sum_{i=1}^{3} \frac{1}{\det(A)}[-A_{12}\partial_x\hat{\phi}_i + A_{11}\partial_y\hat{\phi}_i] = -\frac{A_{12}}{\det(A)}\sum_{i=1}^{3}\partial_x\hat{\phi}_i - \frac{A_{11}}{\det(A)}\sum_{i=1}^{3}\partial_y\hat{\phi}_i = -0 + 0 = 0 \\
\sum_{i=1}^{3} (x_i - x)\partial_x\phi_i(x,y) &= \sum_{i=1}^{3} [A_{11}(\hat{x} - \hat{x}_i) + A_{12}(\hat{y} - \hat{y}_i)]\frac{1}{\det(A)}[A_{22}\partial_x\hat{\phi}_i - A_{21}\partial_y\hat{\phi}_i] \\
&= \frac{1}{\det(A)}\left[\sum_{i=1}^{3} A_{11}A_{22}(\hat{x} - \hat{x}_i)\partial_x\hat{\phi}_i + \cdots\right] = \frac{1}{\det(A)}\left[A_{11}A_{22} - A_{21}A_{12}\right] = 1 \\
\sum_{i=1}^{3} (y_i - y)\partial_x\phi_i(x,y) &= \frac{1}{\det(A)}\sum_{i=1}^{3} [A_{21}(\hat{x} - \hat{x}_i) + A_{22}(\hat{y} - \hat{y}_i)][A_{22}\partial_x\hat{\phi}_i - A_{21}\partial_y\hat{\phi}_i] \\
&= \frac{1}{\det(A)}\left[A_{21}A_{22}\sum_{i=1}^{3}(\hat{x} - \hat{x}_i)\partial_x\hat{\phi}_i + \cdots\right] = \frac{1}{\det(A)}\left[A_{21}A_{22} - A_{21}A_{22}\right] = 0 \\
\sum_{i=1}^{3} (x_i - x)\partial_y\phi_i(x,y) &= \frac{1}{\det(A)}\sum_{i=1}^{3} [A_{11}(\hat{x} - \hat{x}_i) + A_{12}(\hat{y} - \hat{y}_i)][-A_{12}\partial_x\hat{\phi}_i + A_{11}\partial_y\hat{\phi}_i] \\
&= \frac{1}{\det(A)}\left[-A_{11}A_{12}\sum_{i=1}^{3}(\hat{x} - \hat{x}_i)\partial_x\hat{\phi}_i + \cdots\right] = \frac{1}{\det(A)}\left[-A_{11}A_{12} + A_{11}A_{12}\right] = 0 \\
\sum_{i=1}^{3} (y_i - y)\partial_y\phi_i(x,y) &= \frac{1}{\det(A)}\sum_{i=1}^{3} [A_{21}(\hat{x} - \hat{x}_i) + A_{22}(\hat{y} - \hat{y}_i)][-A_{12}\partial_x\hat{\phi}_i + A_{11}\partial_y\hat{\phi}_i] \\
&= \frac{1}{\det(A)}\left[-A_{21}A_{12}\sum_{i=1}^{3}(\hat{x} - \hat{x}_i)\partial_x\hat{\phi}_i + \cdots\right] = \frac{1}{\det(A)}\left[-A_{21}A_{12} + A_{11}A_{22}\right] = 1
\end{aligned}
$$

Thus, we just need to show that all of the formulas hold for the reference triangle.

$$\begin{aligned}
\sum_{i=1}^{3} \hat{\phi}_i(x,y) &= 1 - x - y + x + y = 1 \\
\sum_{i=1}^{3} (x_i - x)\hat{\phi}_i(x,y) &= -x(1-x-y) + (1-x)x + (-x)(y) = -x + x^2 + xy + x - x^2 - xy = 0 \\
\sum_{i=1}^{3} (y_i - y)\hat{\phi}_i(x,y) &= -y(1-x-y) + (-y)x + (1-y)(y) = -y + xy + y^2 + -xy + y - y^2 = 0 \\
\sum_{i=1}^{3} \partial_x \hat{\phi}_i(x,y) &= -1 + 1 + 0 = 0 \\
\sum_{i=1}^{3} \partial_y \hat{\phi}_i(x,y) &= -1 + 0 + 1 = 0 \\
\sum_{i=1}^{3} (x_i - x)\partial_x \hat{\phi}_i(x,y) &= -x(-1) + (1-x)1 + (-x)0 = x + 1 - x = 1 \\
\sum_{i=1}^{3} (y_i - y)\partial_x \hat{\phi}_i(x,y) &= -y(-1) + (-y)1 + (1-y)(0) = y - y = 0 \\
\sum_{i=1}^{3} (x_i - x)\partial_y \hat{\phi}_i(x,y) &= -x(-1) + (1-x)(0) + (-x)(1) = x - x = 0 \\
\sum_{i=1}^{3} (y_i - y)\partial_y \hat{\phi}_i(x,y) &= -y(-1) + (-y)(0) + (1-y)(1) = y + 1 - y = 1
\end{aligned}$$

So we're good and everything works.

```
from dolfin import *
from math import *
import numpy as np
from operator import truediv

# Define source term
class Source(Expression):
        def eval(self, values, x):
                values[0] = 0.0

# Define exact solution
class Exact(Expression):
        def eval(self, values, x):
                r = hypot(x[0],x[1])
                theta = atan2(x[1],x[0])
                if theta < 0:
                        theta = theta + 2*pi
                values[0] = (r ** (2./3))*sin(2./3*theta)

# Dirichlet boundary condition
class DirichletBoundary(SubDomain):
        def inside(self, x, on_boundary):
                return on_boundary

marking = input('Marking Strategy(Uniform(1), Local(2), Dorfler(3)) ')
f = Source()
uexact = Exact()

max_num_element = 5000
num_element = 0

# degree of the elements
deg = 1

tol = 0.15
theta = 0.8
errors = []

# initial mesh
mesh = Mesh('l-shape-mesh.xml')

while num_element < max_num_element:
        num_element = mesh.num_cells()

        # Define function space
        Vh = FunctionSpace(mesh, 'Lagrange', deg)
        bc = DirichletBC(Vh, Exact(), DirichletBoundary())

        # Define variational problem
        v = TestFunction(Vh)
        u = TrialFunction(Vh)
        a = inner(grad(u),grad(v))*dx
        F = f*v*dx

        # Compute solution
        uh = Function(Vh)
        solve(a == F, uh, bc)

        # Compare with exact solution interpolated into high degree space
        Vex = FunctionSpace(mesh, 'Lagrange', deg+2)
        ui = interpolate(uexact, Vex)
        err = ui - interpolate(uh, Vex)

        # Compute error indicator
        h = CellSize(mesh)
        n = FacetNormal(mesh)
        DG0 = FunctionSpace(mesh, 'DG', 0)
```

```
        w = TestFunction(DG0)
        eta2 = assemble((h ** 2) * w * ((div(grad(uh)) + f) ** 2) * dx \
        + avg(w) * avg(h) * (jump(grad(uh),n) ** 2) / 2. * dS)
        eta2 = eta2.array()

        max_eta2 = max(eta2)
        sum_eta2 = sum(eta2)

        # Compute norms of the error
        L2err = sqrt(assemble(err*err*dx))
        H1err = sqrt(assemble(inner(grad(err),grad(err))*dx))
        EI = sqrt(sum(eta2))
        errors.append([num_element, L2err, H1err, EI])

        if sum_eta2 < tol ** 2:
                break

        # Adaptive mesh refinement
        marked = np.zeros(eta2.size, dtype='bool')

        if marking == 1:
                marked = marked + 1
        elif marking == 2:
                marked = marked + (eta2 >= theta**2*max_eta2)
        elif marking == 3:
                while eta2.dot(marked) < (theta**2)*(sum_eta2):
                        # print " " + str(eta2.dot(marked))
                        # print eta2.dot(marked) - (theta**2)*(sum_eta2)
                        tempMax = max(eta2*(1-marked))
                        ind = np.where(eta2 == tempMax)
                        marked[ind] = 1
                # print " " + str(eta2.dot(marked))
                # print eta2.dot(marked) - (theta**2)*(sum_eta2)

        cells_marked = CellFunction('bool', mesh)
        cells_marked.array()[:] = marked

        mesh = refine(mesh, cells_marked);
        fig = plot(mesh);
        fig.write_pdf('MeshRefine' + str(marking) + 'NumElt' + str(num_element));

print "\n Num_element   L2 error        H1 error        EI              L2 rate H1 rate\n"
print "  {:7d}  {:4.2e} {:4.2e} ".format( \
        errors[0][0], errors[0][1], errors[0][2], errors[0][3])
for i in range(1,len(errors)):
        element_ratio = np.log(truediv(errors[i][0],errors[i-1][0]))/2
        print "  {:7d}  {:4.2e} {:4.2e} {:4.2e} {:5.2f}   {:5.2f}".format( \
        errors[i][0], errors[i][1], errors[i][2], errors[i][3],\
        np.log(errors[i-1][1]/errors[i][1])/element_ratio, \
        np.log(errors[i-1][2]/errors[i][2])/element_ratio)
```

Boundary Conditions from Variational Formulation

Matt Charnley

April 4, 2016

We begin with the variational problem: Find $u \in V$ so that for all $v \in V$,

$$\int_0^1 u'(x)v'(x)\ dx + \int_0^1 u(x)v(x)\ dx = \int_0^1 f(x)v(x)\ dx - 2v(0)$$

where

$$V = \{v : ||v||_{L^2} + ||v'||_{L^2} < \infty, \quad v(1) = 0\}.$$

After integrating by parts and using the conditions on the space V, we reduce to the equation

$$\int_0^1 (-u''(x) + u(x))v(x)\ dx - u'(0)v(0) = \int_0^1 f(x)v(x)\ dx - 2v(0)$$

and we want to extract the PDE and boundary conditions from this. We can start by choosing

$$v \in H_0^1 = V \cap \{v(0) = 0\} \subset V$$

which are functions v in V that vanish at both endpoints.

If we plug in such a function, we then see that the second term vanishes, giving that

$$\int_0^1 (-u''(x) + u(x))v(x)\ dx = \int_0^1 f(x)v(x)\ dx$$

which holds for every such $v \in H_0^1$. However, the H_0^1 space is enough to say that if such a relation holds for all $v \in H_0^1$, then we have that

$$-u''(x) + u(x) = f(x)$$

in the entire domain $(0, 1)$, which holds whether we integrate against a function in H_0^1, or a function in V. Our original saying that, for all $v \in V$, we have that

$$\int_0^1 (-u''(x) + u(x))v(x)\ dx - u'(0)v(0) = \int_0^1 f(x)v(x)\ dx - 2v(0)$$

reduces, since we know $-u'' + u = f$ to just say that

$$u'(0)v(0) = 2v(0)$$

Using $v(x) = 1 - x$ then gives that $u'(0) = 2$.

Finally, the other boundary condition comes from the definition of the space V, i.e., that $u(1) = 0$. Therefore we have the ODE

$$-u'' + u = f \qquad u(1) = 0 \qquad u'(0) = 2$$

A.5 Fall 2016 - Math 501 and 503

This section contains the following documents from my time running the Math 501 and 503 problem sessions:

1. A problem set from the middle of the semester, containing written qualifying exam problems
2. The last problem set of the semester, containing a topic review to help them prepare for the finals

Week 5 Qual Problems

Matt Charnley

November 1, 2016

1 Complex Analysis

1. Fall 2016: Complex #1. Use a contour integral to evaluate

$$\int_0^{2\pi} \frac{d\theta}{(2+\cos\theta)^2}.$$

2. Spring 2015: Complex #2. Use contour integration to evaluate the integral

$$\int_0^{2\pi} \frac{d\theta}{a+\sin\theta} \qquad a > 1$$

3. Fall 2014: Complex #1. Use contour integration to show that for all $a > 0$,

$$\int_{-\infty}^{\infty} \frac{\cos ax}{1+x^2}\, dx = \pi e^{-a}.$$

4. Spring 2014: Day 2 #3. Calculate the indefinite integral

$$\int_{-\infty}^{\infty} \frac{\cos(x)}{(x^2+1)(x^2+9)}\, dx$$

   using residues and contour integration. Be sure to justify all limits that you take in the calculation.

5. Spring 2014: Day 1 #9. Suppose that $f(z)$ is a meromorphic function on the extended complex plane (including $z = \infty$) with only two poles: $z = -1$ is a pole of order 1 with principal part $\frac{1}{z+1}$, and $z = 2$ is a pole of order 3 with principal part $\frac{2}{z-2} + \frac{4}{(z-2)^3}$. Suppose further that $f(0) = 1$.

   (a) Determine $\int_{|z|=4} f(z)\, dz$.

   (b) Determine the number of solutions to $f(z) = 1$ in the extended complex plane.

   (c) Determine $f(z)$ explicitly.

6. Fall 2016: Complex #5. Suppose f and g are holomorphic in a region containing the closed unit disc $\bar{D}$. Suppose f has a simple zero at $z = 0$ and vanishes nowhere else in $\bar{D}$. Let

$$f_t(z) = f(z) + tg(z).$$

   Show that, if $t > 0$ is sufficiently small, then

   (a) $f_t(z)$ has a unique zero in $\bar{D}$.

   (b) If z_t is that zero, then the mapping $t \mapsto z_t$ is continuous.

7. Spring 2014: Day 1 #3. Let $f(z)$ be an analytic function defined in some open neighborhood of the closed disc $|z| \le r$. Assume there exists an $a > 0$ such that $|f(0)| < a$ and $|f(z)| > a$ for all z with $|z| = r$. Prove that $f(z)$ must have a zero in the open disc $|z| < r$.

2 Real Analysis

1. Spring 2016: Real #2. Let $[a, b]$ be a bounded interval, and let m be Lebesgue measure. Let M be a positive real number, and $\{f_n\}$ a sequence of measurable functions on $[a, b]$ for which $\int_a^b |f_n| \, dm \leq M$ for all n. Assume that $f_n \to f$ for m almost every x.
   (a) State Fatou's Lemma.
   (b) Show that $\int_a^b |f| \, dm \leq M$.
   (c) Suppose that $||f_n - f||_1 \to 0$. Prove that for every $\epsilon > 0$, there exists $\delta > 0$ such that if $A \subset [a, b]$ is m-measurable with $m(A) \leq \delta$, then $\int_A |f_n| \leq \epsilon$ for all n.
2. Spring 2016: Real #5. Let m denote Lebesgue measure on $\mathbb{R}$ and m^2 Lebesgue measure on $\mathbb{R}^2$. Let $f \in L^1(\mathbb{R})$.
   (a) Show that $h(x, y) = f(x)f(x + y) \in L^1(\mathbb{R}^2, m^2)$.
   (b) Show that for almost every y, $x \mapsto f(x)f(x + y)$ defines a function in $L^1(\mathbb{R})$.
   (c) Give an example of a function $g \in L^1(\mathbb{R})$ for which $x \mapsto g(x)g(x + y)$ is not in $L^1(\mathbb{R})$ for at least one $y \in \mathbb{R}$.
3. Fall 2015: Real #1.
   (a) Let m denote Lebesgue measure on $\mathbb{R}$. Prove that the subset A of $L^1(m)$ defined by
   $$A := \{f \in L^1(m) : \int_{\mathbb{R}} |f| \, dm \leq 1\}$$
   is closed under pointwise convergence.
   (b) Prove that
   $$B := \{f \in L^1(m) : \int_{\mathbb{R}} |f| \, dm \geq 1\}$$
   is not closed under pointwise convergence.
4. Fall 2015: Real #2.
   (a) For α a real number and $\alpha > -1$, prove that $\int_0^\infty x^\alpha e^{-x} \, dm < \infty$ where dm is Lebesgue measure.
   (b) For $\alpha > -1$ and k a positive integer, prove that
   $$\lim_{k\to\infty} \int_0^k x^\alpha \left(1 - \frac{x}{k}\right)^k \, dm = \int_0^\infty x^\alpha e^{-x} \, dm.$$
5. Spring 2015: Real #1. In this problem $(X, \mathcal{M}, \mu)$ is an arbitrary measure space.
   (a) State the monotone convergence theorem.
   (b) Prove that if $f_n : X \to [0, \infty)$ is measurable and non-negative for each positive integer n, then
   $$\int_X \liminf_{n\to\infty} f_n \, d\mu \leq \liminf_{n\to\infty} \int_X f_n \, d\mu$$
   (c) Give an example showing that the above can be false if the functions f_n can take negative values.

6. Spring 2015: Real #5. Let $(X, \mathcal{M}, \mu)$ be a measure space. Suppose that $f_n,\ g_n,\ h_n \in L^1(X, \mathcal{M}, \mu)$, $n \geq 1$ satisfying the inequalities
$$f_n(x) \leq g_n(x) \leq h_n(x) \qquad \forall\, n \in \mathbb{N},\ x \in X.$$
Let $f(x)$, $g(x)$, $h(x)$ be functions so that
$$\lim_{n\to\infty} f_n(x) = f(x) \qquad \lim_{n\to\infty} g_n(x) = g(x) \qquad \lim_{n\to\infty} h_n(x) = h(x)$$
for almost all $x \in X$. Furthermore, assume that $f(x),\ h(x) \in L^1(X, \mathcal{M}, \mu)$ with
$$\lim_{n\to\infty} \int_X f_n(x)\ d\mu = \int_X f(x)\ d\mu \qquad \text{and} \qquad \lim_{n\to\infty} \int_X h_n(x)\ d\mu = \int_X h(x)\ d\mu.$$
Show that
$$\lim_{n\to\infty} \int_X g_n(x)\ d\mu = \int_X g(x)\ d\mu.$$
Hint: Look at $h_n - g_n$ and $g_n - f_n$.

7. Spring 2014: Day 2 #1. Let m denote Lebesgue measure on $[0, 1]$. Let $\{f_n\}$ be a sequence of Lebesgue Measurable functions on $[0, 1]$ with values in $[0, \infty]$ so that
$$\lim_{n\to\infty} f_n(x) = 0$$
for a.e. x and
$$\int_0^1 f_n(x)\ dm(x) = 1 \qquad \forall\, n.$$
Define $g(x) = \sup_n\{f_n(x)\}$. Show that
$$\int_0^1 g(x)\ dm(x) = \infty.$$

More Qual Problems, Again

Matt Charnley

December 9, 2016

Real Topics

1. Convergence Theorems: Fatou, Monotone, Dominated.
2. Types of convergence: Pointwise, Uniformly, in L^1, in measure.
3. Product measures: Definition and construction.
4. Fubini-Tonelli: Statement, how to use them together.
5. Integration on $\mathbb{R}^n$ and Polar Coordinates: It works like Calc 3.
6. Change of Coordinates: The determinant/Jacobian thing. If he did this.
7. Signed measures: What are they?
8. Mutually Singular, Absolutely continuous measures: How do measures relate to each other? How to prove each of these?
9. Jordan and Hahn Decompositions.
10. Lebesgue-Radon-Nikodym Theorem. Set up, necessary assumptions, uses.
11. Lebesgue Differentiation Theorem, Hardy-Littlewood maximal function. If he gets here/it's on the exam.
12. And then everything from the first midterm.

Real Problems

1. Fall 2015: Real #4.
   (i) Let $[a, b]$ be a closed, bounded interval and $f : [a, b] \to \mathbb{R}$. Given an "epsilon-delta" definition of what it means for f to be "absolutely continuous on $[a, b]$."
   (ii) Assume now that $f : [0, 1] \to \mathbb{R}$ has the property that for all $0 < \epsilon < 1$, the restriction of f to $[\epsilon, 1]$ is absolutely continuous. Assume also that there exists some $p > 2$ so that
   $$\int_0^1 x|f'(x)|^p \, dm < \infty$$
   where m is Lebesgue measure. Prove that $\lim_{x\to 0^+} f(x)$ exists and is finite.
2. Fall 2013: Day 2, #1. Let $f \in L^1(-\infty, \infty)$ and let $h > 0$ be fixed. Prove that
   $$\frac{1}{2h}\int_{-\infty}^{\infty}\int_{x-h}^{x+h} f(y) \, dy \, dx = \int_{-\infty}^{\infty} f(x) \, dx$$
3. Fall 2013: Day 2, #8. Let $k(y) = c \exp(-y^2)$ with c chosen so that $\int_{-\infty}^{\infty} k(y) \, dm(y) = 1$. If $f \in L^1(-\infty, \infty)$, prove that, for every x in the Lebesgue set of f, there holds
   $$\lim_{t\to 0^+}\int_{-\infty}^{\infty} f(x-y)\frac{1}{t}k\left(\frac{y}{t}\right) \, dy = f(x)$$

Complex Topics

1. Conformal maps. Know the basic maps and tricks to putting them together.
2. Schwarz Lemma and applications. What do all automorphisms of the disk look like? How about the upper half plane?
3. Montel's Theorem.
4. Rouché's Theorem.
5. Meromorphic functions and Laurent Series.
6. Complex Logarithm and square roots. Branch cuts.
7. Doing contour integrals with these functions. Evaluating real integrals using contour integration of a complex functions. What contours can you use with a branch cut?
8. Residue Theorem.
9. Casorati-Weierstrass and the Removable Singularities Theorem, analyzing types of singularities.

Complex Problems

1. Spring 2016: Complex #2. Let $\mathbb{D}$ denote the open disc in $\mathbb{C}$ and suppose that $f : \mathbb{D} \to \mathbb{D}$ is holomorphic. Prove that
$$\frac{|f(0)| - |z|}{1 + |f(0)||z|} \leq |f(z)| \leq \frac{|f(0)| + |z|}{1 - |f(0)||z|}$$
2. Fall 2015: Complex #4. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of analytic functions on the open set $U \subset \mathbb{C}$ with the properties:
   (i) $\{f_n\}_{n=1}^{\infty}$ is uniformly bounded on compact sets.
   (ii) The sequence $\{f_n\}_{n=1}^{\infty}$ converges pointwise to a function $f(x)$.

   Prove that f is analytic in U. (Hint: Apply the Lebesgue Dominated Convergence Theorem to a suitable contour integral.)
3. Fall 2013: Day 1, #6. Prove that if u is a bounded harmonic function in $\mathbb{H}$ with $\lim_{x \to x_0, y \to 0} u(x, y) = 0$ for all $x_0 \in \mathbb{R}$, then $u \equiv 0$.
4. Fall 2013: Day 2, #3. Prove that if U is an open connected subset of $\mathbb{C}$ and $\mathbf{f} = \{f_n\}_{n=1}^{\infty}$ is a sequence of holomorphic functions on U such that $Re(f_n(z)) > 0$ for all n and all $z \in U$, then $\mathbf{f}$ has a subsequence $\{f_{n_k}\}_{k=1}^{\infty}$ so that either
   (i) f_{n_k} converges uniformly on compact subsets of U to a holomorphic function $f(z)$.
   (ii) f_{n_k} converges uniformly to ∞ on compact subsets of U.

   Make sure your proof indicates clearly where and how you are using the assumption that U is connected.

A.6 Fall 2017 - Math 421

This section contains the following documents from my teaching assistant assignment for Math 421:

1. A worked out homework exercise that was posted to the students after they were having trouble with these types of problems
2. A description of Sturm-Liouville problems to help with something the students were not understanding correctly
3. A review sheet handed out for the second midterm
4. Images of the whiteboard that was projected to the students during one of my online office hours
5. The handout for a workshop that was run at the end of the semester to help review for the final.

Section 4.4: Example 4

Matt Charnley

October 12, 2017

This example computes the inverse Laplace transform

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\}$$

1 Convolution Method

Let $F(s) = G(s) = \frac{1}{s^2+k^2}$. Then

$$\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} = \mathscr{L}^{-1}\{F(s)G(s)\} = f * g$$

where f and g are the inverse Laplace transforms of F and G respectively, namely

$$f(t) = g(t) = \frac{1}{k}\sin(kt)$$

Then, we need to compute the convolution

$$f * g(t) = \frac{1}{k^2}\int_0^t \sin(kt - k\tau)\sin(k\tau)\ d\tau$$

1.1 Book Method

Use the product to sum formula

$$2\sin A\sin B = \cos(A-B) - \cos(A+B)$$

to get

$$\begin{aligned}
f * g(t) &= \frac{1}{k^2}\int_0^t \sin(kt-k\tau)\sin(k\tau)\ d\tau \\
&= \frac{1}{2k^2}\int_0^t \cos(kt-2k\tau) - \cos(kt)\ d\tau \\
&= \frac{1}{2k^2}\left[\frac{-1}{2k}\sin(kt-2k\tau) - \tau\cos(kt)\right]\Big|_0^t \\
&= \frac{1}{2k^2}\left[\frac{1}{2k}(\sin(kt) - \sin(-kt)) - t\cos(kt)\right] \\
&= \frac{\sin kt - kt\cos kt}{2k^3}
\end{aligned}$$

1.2 Direct Computation

We start with the same convolution integral, and want to use trigonometric identities to make the integral doable.

$$
\begin{aligned}
f * g(t) &= \frac{1}{k^2}\int_0^t \sin(kt - k\tau)\sin(k\tau)\ d\tau \\
&= \frac{1}{k^2}\int_0^t [\sin(kt)\cos(k\tau) - \cos(kt)\sin(k\tau)]\sin(k\tau)\ d\tau \\
&= \frac{\sin kt}{k^2}\int_0^t \cos k\tau \sin k\tau\ d\tau - \frac{\cos kt}{k^2}\int_0^t \sin k\tau \sin k\tau\ d\tau \\
&= \frac{\sin kt}{2k^2}\int_0^t \sin 2k\tau\ d\tau - \frac{\cos kt}{2k^2}\int_0^t 1 - \cos 2k\tau\ d\tau \\
&= \frac{\sin kt}{2k^2}\frac{-1}{2k}\cos 2k\tau\ |_0^t - \frac{\cos kt}{2k^2}[\tau - \frac{1}{2k}\sin 2k\tau]\ |_0^t \\
&= \frac{\sin kt}{4k^3} - \frac{\sin kt \cos 2kt}{4k^3} - \frac{t\cos kt}{2k^2} + \frac{\cos kt \sin 2kt}{4k^3}
\end{aligned}
$$

Now, if we rearrange terms in that last line

$$
f*g(t) = \frac{\sin kt}{4k^3} - \frac{t\cos kt}{2k^2} + \frac{\cos kt \sin 2kt}{4k^3} - \frac{\sin kt\cos 2kt}{4k^3} = \frac{\sin kt}{4k^3} - \frac{t\cos kt}{2k^2} + \frac{1}{4k^3}[\cos kt \sin 2kt - \sin kt \cos 2kt]
$$

We see that we get a term like $\cos A \sin B - \sin A \cos B = \sin B - A$. Therefore, we can rewrite this expression as

$$
f * g(t) = \frac{\sin kt}{4k^3} - \frac{t\cos kt}{2k^2} + \frac{1}{4k^3}\sin kt = \frac{\sin kt}{2k^3} - \frac{t\cos kt}{2k^2}
$$

which is the same as the answer the book got.

2 Derivative Method

We know that if we take a derivative in s of a function with (s^2+k^2) in the denominator, we get an $(s^2+k^2)^2$, which matches the denominator that we are trying to get to. Using the formula for the Laplace transform of $tf(t)$, we can write out 4 Laplace transforms that we know.

$$
\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2} \quad \mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2} \quad \mathscr{L}\{t\sin kt\} = \frac{2sk}{(s^2+k^2)^2} \quad \mathscr{L}\{t\cos kt\} = \frac{s^2-k^2}{(s^2+k^2)^2}
$$

We can manipulate this last one into a slightly nicer form

$$
\frac{s^2-k^2}{(s^2+k^2)^2} = \frac{s^2+k^2-2k^2}{(s^2+k^2)^2} = \frac{1}{s^2+k^2} - \frac{2k^2}{(s^2+k^2)^2}
$$

Now, we can try to figure out how to solve the initial problem from here. We can write

$$\begin{aligned}
\frac{1}{(s^2+k^2)^2} &= -\frac{1}{2k^2}\frac{-2k^2}{(s^2+k^2)^2} \\
&= -\frac{1}{2k^2}\left[\frac{-2k^2}{(s^2+k^2)^2}+\frac{1}{s^2+k^2}-\frac{1}{s^2+k^2}\right] \\
&= -\frac{1}{2k^2}\left[\frac{1}{s^2+k^2}-\frac{2k^2}{(s^2+k^2)^2}\right]+\frac{1}{2k^3}\frac{k}{s^2+k^2}
\end{aligned}$$

Thus, the equality

$$\frac{1}{(s^2+k^2)^2} = -\frac{1}{2k^2}\left[\frac{1}{s^2+k^2}-\frac{2k^2}{(s^2+k^2)^2}\right]+\frac{1}{2k^3}\frac{k}{s^2+k^2}$$

tells us that

$$\begin{aligned}
\mathscr{L}^{-1}\left\{\frac{1}{(s^2+k^2)^2}\right\} &= \mathscr{L}^{-1}\left\{-\frac{1}{2k^2}\left[\frac{1}{s^2+k^2}-\frac{2k^2}{(s^2+k^2)^2}\right]+\frac{1}{2k^3}\frac{k}{s^2+k^2}\right\} \\
&= -\frac{1}{2k^2}\mathscr{L}^{-1}\left\{\frac{1}{s^2+k^2}-\frac{2k^2}{(s^2+k^2)^2}\right\}+\frac{1}{2k^3}\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} \\
&= -\frac{1}{2k^2}t\cos kt+\frac{1}{2k^3}\sin kt
\end{aligned}$$

which is, again, the same formula as the book.

Sturm-Liouville Problems and Separation of Variables

Matt Charnley

November 4, 2017

In order to be able to talk about these problems, we need to review two different ODEs that you learned how to solve in 244 (which are also reviewed at the start of section 12.5)

$$y' + \alpha y = 0 \qquad \Rightarrow \qquad y = Ce^{-\alpha x}$$

and for $y'' + \lambda y = 0$, we will split into three cases based on whether λ is positive, negative or 0.

$$\begin{aligned} y'' + \alpha^2 y = 0 &\qquad \Rightarrow \qquad y = C_1 \cos(\alpha x) + C_2 \sin(\alpha x) &\qquad \lambda > 0 \\ y'' = 0 &\qquad \Rightarrow \qquad y = C_1 x + C_2 &\qquad \lambda = 0 \\ y'' - \alpha^2 y = 0 &\qquad \Rightarrow \qquad y = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x) &\qquad \lambda < 0 \end{aligned}$$

For pretty much everything that shows up in separation of variables, we need to deal exclusively with the second order equations. The first order on shows up in the heat equation, but we are generally using the α or λ from the second order problem to solve the first one, so the hard part is in the second order equation. The idea is the following: I am given an ODE of the form $y'' + \lambda y = 0$ along with boundary conditions at $x = 0$ and $x = L$. For a given λ, if we look at each case above, we have two constants to solve for, C_1 and C_2, and two boundary conditions at 0 and L, so it seems like this should always be doable. However, we want a non-trivial solution, so not setting both C_1 and C_2 equal to zero can put an extra constraint on what α can be. Therefore, we want to find all possible values of λ for which we have a non-zero solution to the problem. Let's look at a couple of examples and see what happens.

1 Example 1

Let's consider the following problem

$$y'' + \lambda y = 0 \qquad\qquad y(0) = 0 \quad y(L) = 0$$

In order to figure out all possible λ, we need to break this into three cases.

(a) Case 1: $\lambda = 0$. For this case, we know that we have the solution

$$y(x) = C_1 x + C_2$$

Now, if $y(0) = 0$, this implies that $C_2 = 0$. If, on top of that, we force $y(L) = 0$, this means that $y(L) = C_1 L + 0 = 0$, or $C_1 = 0$ because $L \neq 0$. Therefore, in this case, for $\lambda = 0$, the only solution we get is the zero solution.

(b) Case 2: $\lambda = -\alpha^2 < 0$. In this case, our solution takes the form

$$y(x) = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x)$$

If we plug $x = 0$ into this, we get that

$$y(0) = C_1 \cosh(0) + C_2 \sinh(0) = C_1$$

which we want to be zero by boundary conditions. Then, if we plug in $x = L$, we get that

$$y(L) = C_2 \sinh(\alpha L)$$

which we want to be zero to meet the boundary condition. We also do not want to pick $C_2 = 0$, because that would again give us the zero solution. Therefore, we need to have that $\sinh(\alpha L) = 0$. However, if you look at the graph or definition of the sinh function, it is only zero at 0, and nowhere else. Therefore, since $\alpha > 0$ and $L \neq 0$, we can never have that $\sinh(\alpha L) = 0$, which means the only way we can meet the second boundary condition is by setting $C_2 = 0$, giving us the trivial solution for $\lambda < 0$. Thus, we also get no interesting solutions in this case.

(c) Case 3: $\lambda = \alpha^2 > 0$. Well, we've failed twice so far, so hopefully this one works. In this case, our solution looks like

$$y(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$$

If we plug in $x = 0$, we get

$$y(0) = C_1 \cos(0) + C_2 \sin(0) = C_1$$

which we want to be zero by the boundary condition. Then, if we plug in $x = L$, we get

$$y(L) = C_2 \sin(\alpha L) = 0$$

which puts us in a similar position as the previous case. However, this time, since we don't want $C_2 = 0$, we need $\sin(\alpha L) = 0$, and thankfully, the sine function is periodic, and hits zero more than just once. In order for $\sin(\alpha L) = 0$, we need $\alpha L = n\pi$, or $\alpha = \frac{n\pi}{L}$ for any integer n. Then, for each n, we get an eigenvalue $\lambda_n = \frac{n^2\pi^2}{L^2}$ and an eigenfunction $y_n = \sin\left(\frac{n\pi x}{L}\right)$ which satisfies

$$y_n'' + \lambda_n y_n = 0 \qquad y_n(0) = 0 \quad y_n(L) = 0$$

Therefore, for this problem, the full set of eigenvalues is $\lambda_n = \frac{n^2\pi^2}{L^2}$ for $n = 1, 2, ...$ and the corresponding eigenfunction are $y_n = \sin\left(\frac{n\pi x}{L}\right)$.

2 Example 2

Another problem

$$y'' + \lambda y = 0 \qquad y'(0) = 0 \quad y(L) = 0$$

In order to figure out all possible λ, we need to break this into three cases.

(a) Case 1: $\lambda = 0$. For this case, we know that we have the solution

$$y(x) = C_1 x + C_2$$

Now, if $y'(0) = 0$, this implies that $C_1 = 0$. If, on top of that, we force $y(L) = 0$, this means that $y(L) = 0 + C_2 = 0$, or $C_2 = 0$. Therefore, in this case, for $\lambda = 0$, the only solution we get is the zero solution.

(b) Case 2: $\lambda = -\alpha^2 < 0$. In this case, our solution takes the form

$$y(x) = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x)$$

If we plug $x = 0$ into the derivative, we get that

$$y'(0) = C_1 \alpha \sinh(0) + C_2 \alpha \cosh(0) = C_2 \alpha$$

which we want to be zero by boundary conditions, so $C_2 = 0$. Then, if we plug in $x = L$, we get that

$$y(L) = C_1 \cosh(\alpha L)$$

which we want to be zero to meet the boundary condition. We also do not want to pick $C_1 = 0$, because that would again give us the zero solution. Therefore, we need to have that $\cosh(\alpha L) = 0$. However, if you look at the graph or definition of the cosh function, it is never equal to zero. Therefore, the only way we can meet the second boundary condition is by setting $C_1 = 0$, giving us the trivial solution for $\lambda < 0$. Thus, we also get no interesting solutions in this case.

(c) Case 3: $\lambda = \alpha^2 > 0$. As you probably guessed, this is the one that works here too. In this case, our solution looks like

$$y(x) = C_1 \cos(\alpha x) + C_2 \sin(\alpha x)$$

If we plug $x = 0$ into the derivative, we get

$$y'(0) = -C_1 \alpha \sin(0) + C_2 \alpha \cos(0) = C_2$$

which we want to be zero by the boundary condition. Then, if we plug in $x = L$, we get

$$y(L) = C_1 \cos(\alpha L) = 0$$

which puts us in a similar position as the previous case. However, this time, since we don't want $C_1 = 0$, we need $\cos(\alpha L) = 0$, which is again periodic. To get $\cos(\alpha L) = 0$, we need $\alpha L = (n + \frac{1}{2})\pi$, so that

$$\alpha_n = \frac{\left(n + \frac{1}{2}\right)\pi}{L}$$

Then, for each n, we get an eigenvalue $\lambda_n = \frac{(n+\frac{1}{2})^2\pi^2}{L^2}$ and an eigenfunction $y_n = \cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)$ which satisfies

$$y_n'' + \lambda_n y_n = 0 \qquad y_n'(0) = 0 \quad y_n(L) = 0$$

Therefore, for this problem, the full set of eigenvalues is $\lambda_n = \frac{(n+\frac{1}{2})^2\pi^2}{L^2}$ for $n = 1, 2, ...$ and the corresponding eigenfunction are $y_n = \cos\left(\frac{(n+\frac{1}{2})\pi x}{L}\right)$.

3 Example 3

More problems:

$$y'' + \lambda y = 0 \qquad y(-1) = 0 \quad y'(1) - \frac{1}{2}y(1) = 0$$

(a) Case 1: $\lambda = 0$. For this case, we know that we have the solution

$$y(x) = C_1 x + C_2$$

Now, if $y(-1) = 0$, this implies that $-C_1 + C_2 = 0$. If, on top of that, we force $y'(1) - \frac{1}{2}y(1) = 0$, this means that $y'(1) - \frac{1}{2}y(1) = C_1 - \frac{1}{2}C_1 - \frac{1}{2}C_2$, or, again, $C_1 - C_2 = 0$. Thus, we can pick $C_1 = C_2 = 1$ to get a non-trivial solution, $y(x) = x + 1$. Therefore, in this case, for $\lambda = 0$, we get the eigenfunction $y(x) = x + 1$.

(b) Case 2: $\lambda = -\alpha^2 < 0$. In this case, our solution takes the form

$$y(x) = C_1 \cosh(\alpha x) + C_2 \sinh(\alpha x)$$

If we plug $x = -1$ into the derivative, we get that

$$y(-1) = C_1 \cosh(-\alpha) + C_2 \sinh(-\alpha) = 0$$

Next, we check the other boundary condition

$$y'(1) - \frac{1}{2}y(1) = C_1\alpha\sinh(\alpha) + C_2\alpha\cosh(\alpha) - \frac{1}{2}C_1\cosh(\alpha) - \frac{1}{2}C_2\sinh(\alpha) = 0$$

Then we are left to see for which values of α there is a solution to

$$\begin{aligned} C_1\cosh(\alpha) - C_2\sinh(\alpha) &= 0 \\ C_1\left(\alpha\sinh(\alpha) - \frac{1}{2}\cosh(\alpha)\right) + C_2\left(\alpha\cosh(\alpha) - \frac{1}{2}\sinh(\alpha)\right) &= 0 \end{aligned}$$

It takes some work to solve this out, but it turns out the only solution here occurs when $\alpha = 0$, so there are no non-trivial solutions in this case.

(c) Case 3: $\lambda = \alpha^2 > 0$. In this case, our solution looks like

$$y(x) = C_1\cos(\alpha x) + C_2\sin(\alpha x)$$

Plugging into the boundary conditions, we get that

$$y(-1) = C_1\cos(-\alpha) + C2\sin(-\alpha) = 0$$

and

$$y'(1) - \frac{1}{2}y(1) = -C_1\alpha\sin(\alpha) + C_2\alpha\cos(\alpha) - \frac{1}{2}C_1\cos(\alpha) - \frac{1}{2}C_2\sin(\alpha) = 0$$

which reduces to the system of equations

$$\begin{aligned} C_1\cos(\alpha) - C_2\sin(\alpha) &= 0 \\ C_1\left(-\frac{1}{2}\cos(\alpha) - \alpha\sin(\alpha)\right) + C_2\left(\alpha\cos(\alpha) - \frac{1}{2}\sin(\alpha)\right) &= 0 \end{aligned}$$

In order to have a non-zero solution, the determinant of the coefficient matrix (in terms of the constants C) must vanish, which means that

$$\alpha\cos(\alpha)^2 - \frac{1}{2}\cos(\alpha)(\sin(\alpha) - \alpha\sin(\alpha)^2 - \frac{1}{2}\sin(\alpha)(\cos(\alpha) = 0$$

which can be simplified to

$$\alpha(\cos^2(\alpha) - \sin^2(\alpha)) - \sin(\alpha)\cos(\alpha) = 0$$

Using double angle formulas, this becomes

$$\alpha cos(2\alpha) - \frac{1}{2}\sin(2\alpha) = 0$$

or

$$\alpha = \frac{1}{2}\tan(2\alpha)$$

As discussed in the textbook, this has an infinite sequence of solutions with $\alpha_n \to \infty$, and each of them gives rise to an eigenfunction. These are impossible to find by hand, but we know that they exist. These eigenfunctions and eigenvalues will all solve the same problem as in the above examples.

Thus for this problem, we get both an eigenvalue at 0 with corresponding linear solution $y_0(x) = x + 1$ and a sequence of positive eigenvalues with corresponding eigenfunctions that we can't find by hand.

4 Discussion

In each case before, we saw that the sine and cosine functions turned out to be the largest quantity of solutions we found. This is no coincidence. While it is possible to get some eigenfunctions with the linear solution or the sinh/cosh functions, you can not get infinitely many. This is basically what is stated in the theorem about Sturm Liouville problems in that the sequence of eigenvalues converges to $+\infty$, which means there can only be finitely many less than 0. Thus, for any of these reqular problems, you may see a few particular solutions with non-positive eigenvalues, but you will then always see an infinite family of solutions (with a sequence of positive eigenvalues) that involve sines and cosines.

Separation of Variables

The main way this is used (at least for this class) is in separation of variables problems. The idea is that once you separate variables, you are looking for solutions to certain Sturm-Liouville problems with boundary conditions that correspond to the boundary conditions of your initial problem. In the end, you will generally need to use generalized Fourier Series to find the final solution, because this sequence of functions you generate from the Sturm-Liouville problem will be an orthogonal set of functions, which you can then use in a Fourier-type expansion. The set from example 1 results in a standard sine series expansion, etc.

Exam 2 Review Problems

Matt Charnley

November 28, 2017

13. f is defined as the following function

$$f(x) = \begin{cases} 0 & -2 < x < 0 \\ x & 0 < x < 1 \\ 0 & 1 < x < 2 \end{cases}$$

To compute the Fourier Series of f, we just need to calculate the coefficients based on the formulas in the book.

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)\ dx \qquad a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\ dx \qquad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\ dx$$

where in this case we have $L = 2$. We then calculate

$$a_0 = \frac{1}{2}\int_{-2}^{2} f(x)\ dx = \frac{1}{2}\int_0^1 x\ dx = \frac{1}{2}\frac{x^2}{2}\ |_0^1 = \frac{1}{4}$$

$$\begin{aligned} a_n &= \frac{1}{2}\int_0^1 x\cos\left(\frac{n\pi x}{2}\right)\ dx \\ &= \frac{1}{2}\left[x\sin\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\right] |_0^1 - \frac{1}{n\pi}\int_0^1 \sin\left(\frac{n\pi x}{2}\right)\ dx \\ &= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\frac{2}{n\pi}\cos\left(\frac{n\pi x}{2}\right) |_0^1 \\ &= \frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{(n\pi)^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right) \end{aligned}$$

$$\begin{aligned} b_n &= \frac{1}{2}\int_0^1 x\sin\left(\frac{n\pi x}{2}\right)\ dx \\ &= \frac{1}{2}\left[-x\cos\left(\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\right] |_0^1 + \frac{1}{n\pi}\int_0^1 \cos\left(\frac{n\pi x}{2}\right)\ dx \\ &= \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{1}{n\pi}\frac{2}{n\pi}\sin\left(\frac{n\pi x}{2}\right) |_0^1 \\ &= \frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right) \end{aligned}$$

Therefore the Fourier Series of f is

$$FS(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\left(\frac{n\pi x}{2}\right) + b_n\sin\left(\frac{n\pi x}{2}\right)$$

which evaluates to

$$FS(x) = \frac{1}{8} + \sum_{n=1}^{\infty}\left(\left[\frac{1}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{2}{(n\pi)^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)\right]\cos\left(\frac{n\pi x}{2}\right) + \left[\frac{1}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{2}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)\right)$$

The graphs are shown on the back pages. This function is periodic with period 4.

14. For this problem, we consider the same f restricted to just $(0, 2)$. We want to compute the Fourier Sine Series. We do that by

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

with $L = 2$ for this particular f. This gives

$$b_n = \frac{2}{2}\int_0^1 x\sin\left(\frac{n\pi x}{L}\right)\,dx$$

and, up to the multiple in front, this is the same thing we calculated for the b_n in problem 13. Thus, we have

$$b_n = \frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)$$

and the sine series expansion is

$$FS_S(x) = \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]\sin\left(\frac{n\pi x}{2}\right)$$

The graph is at the end. This function is also periodic with period 4.

15. Now, we want to do the cosine series for the same function. We calculate this by

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)\,dx$$

and, exactly as we saw before, since f is only non-zero on $(0, 1)$, these will turn out to be exactly double the coefficients that we calculated in problem 13. Thus we get

$$a_0 = \frac{1}{2} \qquad a_n = \frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)$$

and the Fourier Cosine Series expansion is

$$FS_C(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\left[\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right) + \frac{4}{(n\pi)^2}\left(\cos\left(\frac{n\pi}{2}\right) - 1\right)\right]\cos\left(\frac{n\pi x}{2}\right)$$

Graphs are again at the end. This function is periodic with period 4.

16. Now, something a little different. We want to do the periodic extension. These are calculated by the formulas

$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx \qquad a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{2n\pi x}{L}\right)\,dx \qquad b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{2n\pi x}{L}\right)\,dx$$

with $L = 2$. We calculate all of these coefficients in the same way. Except these turn out a little nicer.

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = \int_0^1 x\,dx = \frac{x^2}{2}\,\Big|_0^1 = \frac{1}{2}$$

$$
\begin{aligned}
a_n &= \frac{2}{2}\int_0^1 x\cos\left(\frac{2n\pi x}{2}\right)\,dx = \int_0^1 x\cos(n\pi x)\,dx \\
&= x\frac{1}{n\pi}\sin(n\pi x)\mid_0^1 - \int_0^1 \frac{1}{n\pi}\sin(n\pi x)\,dx \\
&= 0 + \frac{1}{(n\pi)^2}\cos(n\pi x)\mid_0^1 \\
&= \frac{1}{(n\pi)^2}((-1)^n - 1)
\end{aligned}
$$

$$
\begin{aligned}
b_n &= \frac{2}{2}\int_0^1 x\sin\left(\frac{2n\pi x}{2}\right)\,dx = \int_0^1 x\sin(n\pi x)\,dx \\
&= -x\cos(n\pi x)\frac{1}{n\pi}\mid_0^1 + \int_0^1 \frac{1}{n\pi}\cos(n\pi x)\,dx \\
&= \frac{(-1)^{n+1}}{n\pi} + \frac{1}{n\pi}^2\sin(n\pi x)\mid_0^1 \\
&= \frac{(-1)^{n+1}}{n\pi}
\end{aligned}
$$

So then, the Fourier Series is

$$FS_P(x) = \frac{1}{4} + \sum_{n=1}^{\infty}\frac{(-1)^n - 1}{(n\pi)^2}\cos(n\pi x) + \frac{(-1)^{n+1}}{n\pi}\sin(n\pi x)$$

This is a periodic function of period 2. See the graphs at the end.

17. For complex Fourier Series, we use the formula

$$c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}\,dx$$

For this function in particular, we compute

$$
\begin{aligned}
c_n &= \frac{1}{4}\int_0^1 xe^{-in\pi x/2}\,dx && (1)\\
&= \frac{-2}{in\pi}xe^{-in\pi x/2}\mid_0^1 + \frac{2}{in\pi}\int_0^1 e^{-in\pi x/2}\,dx && (2)\\
&= \frac{-2}{in\pi}e^{-in\pi/2} + \frac{4}{(n\pi)^2}e^{-in\pi x/2}\mid_0^1 && (3)\\
c_n &= \frac{-2}{in\pi}e^{-in\pi/2} + \frac{4}{(n\pi)^2}(e^{-in\pi/2} - 1) && (4)
\end{aligned}
$$

However, this formula doesn't work for c_0, so we individually compute

$$c_0 = \frac{1}{4}\int_0^1 x\,dx = \frac{1}{8}$$

With this definition of c_n, we then have that the complex fourier series of f is

$$FS_{\mathbb{C}}(x) = \sum_{n=-\infty}^{\infty} c_n e^{in\pi x/2}$$

18. Check the Sturm-Liouville document already on Sakai for details on this problem. That should give you enough to go on. You should get eigenvalues of 0 and n^2 for $n = 1, 2, ...$ and corresponding eigenfunctions of constants and $\cos(nx)$.

19. For characterizing these types of PDE's, all we care about are the second order terms, i.e., those with two derivatives. The other terms do not matter. Thus, we compute $B^2 - 4AC$ for this problem and get $1 - 4 \cdot 4 \cdot 3 < 0$, so this is an elliptic equation.

20. In order to do separation of variables, we make the assumption that $u(x,t) = X(x)T(t)$. Plugging this into the equation gives us

$$XT' - X'T = XT$$

Dividing both sides by XT, we get

$$\frac{T'}{T} - \frac{X'}{X} = 1$$

and rearranging the terms, we see

$$fracT'T = \frac{X'}{X} + 1 = -\lambda$$

where we can now use the separation constant because the left side only depends on t, and the right side only depends on x. This gives us two separate ODEs to solve

$$\frac{T'}{T} = -\lambda \qquad \rightarrow \qquad T' + \lambda T = 0$$

$$\frac{X'}{X} + 1 = -\lambda \qquad \rightarrow \qquad X' + (\lambda + 1)X = 0$$

These two ODEs have easy solutions, namely

$$T(t) = Ce^{-\lambda t} \qquad\qquad X(x) = Ce^{-(\lambda+1)x}$$

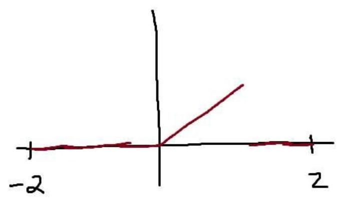
Therefore, the separated solutions look like

$$u(x,t) = Ce^{-\lambda t}e^{-(\lambda+1)x}$$

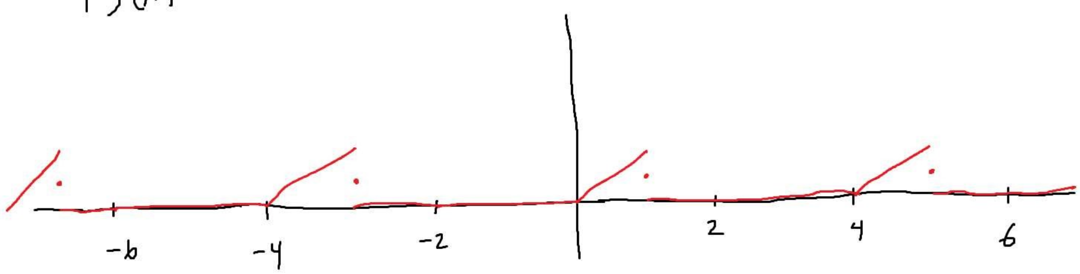
Graphs

Note: All lines should be straight.

13| f(x)

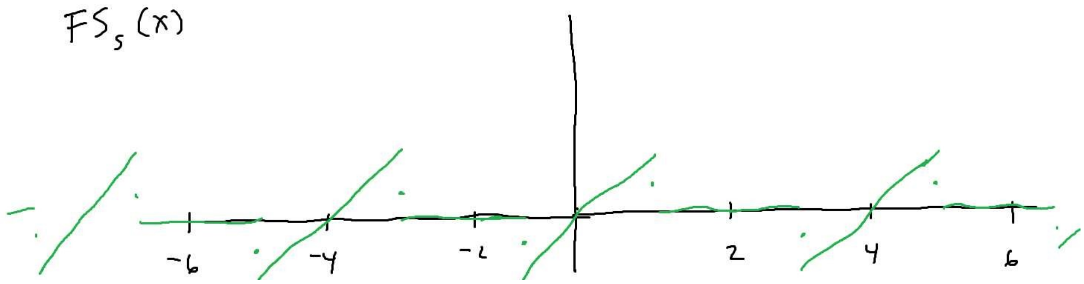


FS(x)



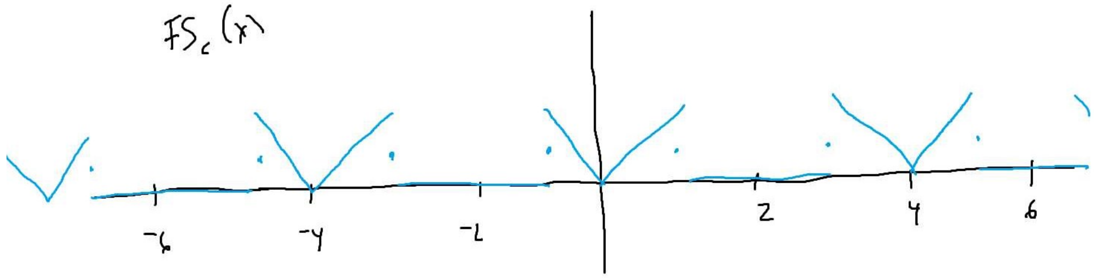
14| Sine Series

$FS_s(x)$



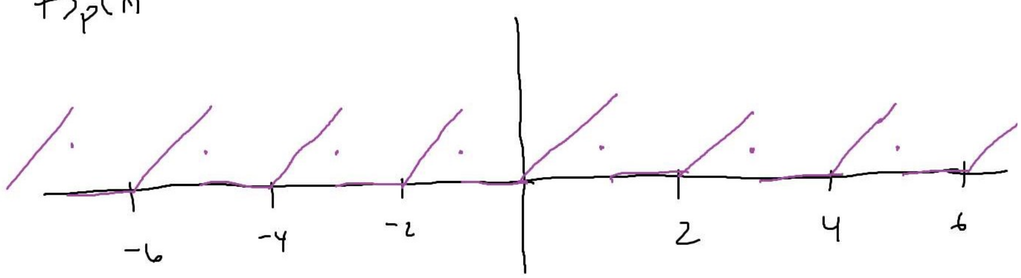
5 Cosine Series

$FS_c(x)$



6 Periodic Extension

$FS_p(x)$

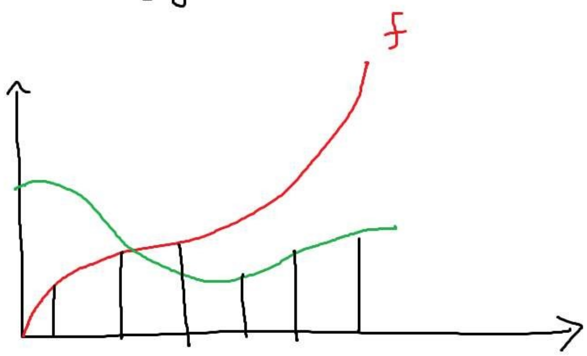


<u>Convolutions</u>

f, g defined on $[0, \infty)$ then

$$f * g(t) = \int_0^t f(t-\tau)g(\tau)\,d\tau$$

$$f * g(1) = \int_0^1 f(1-\tau)\,g(\tau)\,d\tau$$



$f(t) = e^t \qquad g(t) = t^2$

$\int^t$

$$f*g(t) = \int_0 e^{t-\tau}\tau^2\, d\tau$$

4.4 #26

$$\mathcal{L}\left\{\int_0^t \tau \sin\tau\, d\tau\right\}$$

$f(t) = t \quad g(t) = \sin t$

$$\underline{f*g(t) = \int_0^t (t-\tau)\sin\tau\, d\tau}$$

$$\mathcal{L}\{f*g(t)\} = \mathcal{L}\{f\}\cdot\mathcal{L}\{g\}$$

$$\mathcal{L}\{f\} \quad \mathcal{L}\{g\}$$

$$f*g$$

$$\mathcal{L}\left\{\underbrace{\int_0^t (t-\tau)\sin\tau\, d\tau}_{\shortparallel}\right\} = \mathcal{L}\{t\}\cdot\mathcal{L}\{\sin t\}$$

$$\frac{1}{s^2}\cdot\frac{1}{s^2+1}$$

$$\mathcal{L}\left\{\int_0^t t\sin\tau \, d\tau\right\} - \underbrace{\mathcal{L}\left\{\int_0^t \tau\sin\tau \, d\tau\right\}}_{\text{Answer}}$$

$$\mathcal{L}\left\{t\int_0^t \sin\tau \, d\tau\right\}$$

$$= \mathcal{L}\left\{-t\cos t + t\right\}$$

$$= \frac{d}{ds}\mathcal{L}\left\{\cos t\right\} + \frac{1}{s^2}$$

$$\boxed{\mathcal{L}\left\{t f(t)\right\} = -\frac{d}{ds}F(s)}$$

$$= \frac{d}{ds}\left(\frac{s}{s^2+1}\right) + \frac{1}{s^2}$$

$$= \frac{s^2+1-s(2s)}{(s^2+1)^2} + \frac{1}{s^2}$$

$$= \frac{1-s}{(s^2+1)^2} + \frac{1}{s^2}$$

$$\frac{1}{s^2(s^2+1)} = \frac{1-s^2}{(s^2+1)^2} + \frac{1}{s^2} - A$$

<u>Direct way</u>

$$\mathcal{L}\left\{ \int_0^t \tau \sin \tau \, d\tau \right\} = \mathcal{L}\{1\} \cdot \mathcal{L}\{t \sin t\}$$

$$\int_0^t \tau \sin \tau \, d\tau = f * g = \int_0^t f(t-\tau) g(\tau) \, d\tau$$

So choose: $f(t) = 1 \quad g(t) = t \sin t$

421 Workshop 1

December 13, 2017

1 Chapter 4: The Laplace Transform

1.1 Introduction

Over the course of chapter 4, we learned many different techniques for computing Laplace Transforms and Inverse Laplace Transforms. For reference, some/most of them are listed here.

1. Computing from the definition of Laplace Transform (4.1)

$$\mathcal{L}[f(t)] = \int_0^\infty e^{-st} f(t)\ dt$$

2. Using linearity from a table of known transforms (4.1)

3. Computing Inverse Transforms using Partial Fractions (4.2)

4. Formulas for computing the transform of a derivative, or the derivative of a transform (4.2, 4.5)

$$\mathcal{L}[f'(t)] = sF(s) - f(0) \qquad \qquad \mathcal{L}[tf(t)] = -\frac{d}{ds}F(s)$$

5. Translation Theorems (4.3)

$$\begin{aligned}\mathcal{L}[f(t-a)\mathcal{U}(t-a)] &= e^{-as}F(s)\\ \mathcal{L}[e^{at}f(t)] &= F(s-a)\\ \mathcal{L}[f(t)\mathcal{U}(t-a)] &= e^{-as}\mathcal{L}[f(t+a)]\end{aligned}$$

6. Convolutions (4.4)

$$\mathcal{L}[f * g(t)] = F(s)G(s) \qquad \text{where} \quad f * g(t) = \int_0^\infty f(\tau)g(t-\tau)\ d\tau$$

7. Periodic functions (4.4)

$$\mathcal{L}[f(t)] = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)\ dt$$

8. Delta functions (4.5)

$$\mathcal{L}[\delta(t-t_0)] = e^{-t_0 s}$$

Use these properties to compute the Laplace Transforms and Inverse Laplace transforms on the following pages. You'll have to decide which method to use.

1.2 Problems

Compute the Laplace Transform for each of the following functions $f(t)$

1. $$f(t) = \begin{cases} t & 0 < t < 5 \\ 3 & t > 5 \end{cases}$$

2. $$f(t) = \begin{cases} 0 & 0 < t < \pi \\ \cos(t) & t > \pi \end{cases}$$

3. $$f(t) = \begin{cases} 2 & 0 < t < 1 \\ t & 1 < t < 3 \\ e^{5t} & t > 3 \end{cases}$$

4. $f(t) = t^4 + 3t^2 + \sin(2t)$
5. $f(t) = (\sin(t) + t^2)\mathcal{U}(t-\pi)$
6. $f(t) = t^2 \cos(t)$
7. $f(t) = e^{2t}(t^2 + 3t)$
8. $f(t) = (1 + e^{3t} - 2e^{-t})\cos(5t)$
9. $f(t) = t^2(e^t + e^{2t})^2$
10. $f(t) = (t^2 - 4t + 1)\mathcal{U}(t-2)$
11. $f(t) = \int_0^t \tau \sin(t-\tau)\ d\tau$
12. $f(t) = t\int_0^t \tau^2 e^{t-\tau}\ d\tau$
13. $f(t) = t^2 \int_0^t \cos(\tau)\ d\tau$
14. $f(t) = te^{3t}\sin(2t)$
15. $f(t) = e^{-2t}t^3\mathcal{U}(t-4)$

Compute the inverse Laplace Transform for each of the following functions $F(s)$.

1. $F(s) = \frac{2s+5}{s^2+9}$
2. $F(s) = \frac{4s-3}{s^2+2s+17}$
3. $F(s) = \frac{s^2+4s-1}{s^3-5s^2+6s}$
4. $F(s) = e^{-s}\frac{3s}{(s+1)^2+4}$
5. $F(s) = \frac{2}{5s+3}$
6. $F(s) = \frac{32}{s^5} + \frac{6}{s^2}$
7. $F(s) = \frac{s+3}{(s^2+1)(s^2-4)}$
8. $F(s) = \frac{4}{(s+3)^2}$
9. $F(s) = \frac{(s+1)^2}{(s+2)^4}$
10. $F(s) = \frac{e^{-2s}}{s^3}$
11. $F(s) = (1-e^{-s})\frac{1}{s^2+1}$
12. $F(s) = \frac{(2s+1)e^{-\pi s}}{(s^2+25)(s^2+1)}$
13. $F(s) = \frac{1}{s(s^2+1)}$
14. $F(s) = \frac{2s+1}{s^2(s^2+4)}$

1.3 ODEs

We also saw how the Laplace Transform can be used to solve ODEs and integral equations. Solve the following differential equations and systems using Laplace Transforms.

1. $y' - 3y = 4\cos(t) \qquad y(0) = 2$
2. $y'' - 2y' + y = 3e^t \qquad y(0) = 1, y'(0) = 3$
3. $y' - 3y = 1 + te^{-t} \qquad y(0) = 0$
4. $y'' - y' = e^t \cos(2t) \quad y(0) = 0, y'(0) = 0$
5. $y' - 2y = f(t) \qquad y(0) = 1$ where
$$f(t) = \begin{cases} 0 & 0 < t < 2 \\ 3 & t > 2 \end{cases}$$
6. $y'' - 4y' - 5y = f(t) \qquad y(0) = 0, y'(0) = 1$ where
$$f(t) = \begin{cases} t & 0 < t < 4 \\ 1 & t > 4 \end{cases}$$
7. $y'' + y = f(t) \qquad y(0) = 2, y'(0) = 1$ where
$$f(t) = \begin{cases} 0 & 0 < t < \pi \\ \sin(t) & t > \pi \end{cases}$$
8. $y'' - 4y = f(t) \qquad y(0) = 3, y'(0) = 0$ where
$$f(t) = \begin{cases} 0 & 0 < t < 1 \\ 2 & 1 < t < 3 \\ 1 & t > 3 \end{cases}$$
9. $f(t) = 3t + \int_0^t f(t-\tau)\ d\tau$
10. $f'(t) - 1 = \sin(t) + \int_0^t f(\tau)\ d\tau \qquad f(0) = 0$
11. $t + 2f(t) = \int_0^t (e^{t-\tau} - e^{\tau - t}) f(\tau)\ d\tau$
12. $y'' - 6y' + 9y = 1 + \delta(t-2) \qquad y(0) = 1, y'(0) = 0$
13. $y'' + 4y = \delta(t - 2\pi) + \delta(t - 4\pi) \qquad y(0) = 0, y'(0) = 0$
14. $x' = 3x - 2y \quad y' = x + 4y \qquad x(0) = 2, y(0) = 1$
15. $2x' + y' - 2x = 1 \quad x' + y' - 3x - 3y = 2 \qquad x(0) = 0, y(0) = 0$
16. $x'' + x - y = 0 \quad y'' + y - x = 0 \quad x(0) = 0, x'(0) = -2, y(0) = 0, y'(0) = -1$

2 Chapter 8: Matrices

2.1 Introduction

2.1.1 Row Reduction

In the first part of chapter 8, we learned about row reduction of matrices, and in later sections, we saw what it allowed us to do in terms of simplifying calculations using matrices.

- Solve a linear system:

  To solve the given system

  $$\begin{aligned} a_1x + b_1y + c_1z &= d_1 \\ a_2x + b_2y + c_2z &= d_2 \\ a_3x + b_3y + c_3z &= d_3 \end{aligned}$$

  we row reduce the augmented matrix

  $$\begin{bmatrix} a_1 & b_1 & c_1 & \mid & d_1 \\ a_2 & b_2 & c_2 & \mid & d_2 \\ a_3 & b_3 & c_3 & \mid & d_3 \end{bmatrix}$$

- Compute the rank of a matrix: The **rank** of a matrix is the number of nonzero rows in its row-reduced form. Equivalently, it is the number of columns in its row-reduced form which have pivots.

- Find the determinant of a matrix by row reducing to triangular form: Row-reduce the matrix, keeping track of your steps to use the properties of determinants under row operations (see below). The determinant of an upper- or lower-triangular matrix is the product of the elements on the diagonal.

- Find the inverse of a matrix: For a 2x2 matrix, you may memorize the formula:

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \tag{1}$$

  For an $n \times n$ matrix A, we can set up the augmented matrix with A on the left and I_n (the $n \times n$ identity matrix) on the right. Row reduce this augmented matrix. If you end up with I_n on the left, then whatever is on the right at that point is A^{-1}. If you end up with a row of zeros, the matrix is **singular**, i.e. not invertible.

2.1.2 Properties of Determinant

- Invertibility: A matrix A is invertible *if and only if* $\det(A) \neq 0$. In other words, both of the following two statements are true:
  - If you already know that A is invertible, then you automatically also know that $\det(A) \neq 0$.
  - If you compute that $\det(A) \neq 0$, then that immediately tells you that A is invertible.

  When A is invertible, $\det(A^{-1}) = \frac{1}{\det(A)}$.
- Determinant of a product: $\det(AB) = \det(A)\det(B)$
- Determinant of the transpose: $\det(A) = \det(A^T)$
- Change in determinant under row operations:
  - Suppose that matrix B matches matrix A, except that two of the rows are swapped. Then

    $$\det(B) = -\det(A) \tag{2}$$
  - Suppose that matrix C matches matrix A, except that one of the rows is multiplied by some number c. Then,

    $$\det(C) = c\det(A)$$
  - Suppose that matrix D is obtained from matrix A by taking one row and adding a multiple of another row to it. Then

    $$\det(D) = \det(A)$$

2.1.3 Eigenvalues and Eigenvectors

- Find eigenvalues and eigenvectors: For a given $n \times n$ matrix A, we say that the nonzero vector K is an **eigenvector** for A with **eigenvalue** λ if

  $$AK = \lambda K \tag{3}$$

  To find the eigenvalues of A, we start by creating a polynomial $p(\lambda)$, which is often called the **characteristic polynomial** of A:

  $$p(\lambda) = \det(A - \lambda I_n)$$

  Then, the eigenvalues of A are the zeros of p. In other words, the eigenvalues of A are the values of λ that solve the **characteristic equation** of A:

$$0 = p(\lambda) = \det(A - \lambda I_n)$$

Then, to find the eigenvectors that correspond to each λ, we reduce the augmented matrix with $A - \lambda I_n$ on the left and a column of 0's on the right, namely $(A - \lambda I_n|0)$.

- Repeated roots: For some matrices, the characteristic polynomial may have repeated roots, ie roots with multiplicity 2 or more. For example, $p(\lambda) = \lambda^2(1-\lambda)$ has $\lambda = 0$ as a root with multiplicity 2. In these cases, it is possible that you will be able to find two or more linearly independent eigenvectors corresponding to the same eigenvalue, up to the multiplicity of that root.

- A note on the zero vector: Suppose I had the vector Z, all of whose entries are 0, ie

$$Z = \begin{pmatrix} 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then, for any matrix A, $AZ = Z$, because the zeros cancel everything out. Furthermore, for any number λ, we also have that $\lambda Z = Z$, again because the zeros cancel everything out. It is for this reason that we don't consider Z an eigenvector: an eigenvector K can have some entries be 0, but it must have at least one entry which is nonzero. The zero vector satisfies $AZ = \lambda Z$ for *every* A and *every* λ, so the fact that it satisfies the equation tells us nothing about A. Furthermore, we want each eigenvector to correspond to *one* eigenvalue. If we allow Z to be an eigenvector, it would have *every* possible λ as an eigenvalue. There are also many other reasons that we don't include Z as a possibility in our definition of an eigenvector, which I won't detail here.

2.2 Computational Problems

For the problems below, either solve the given system or show that no solution exists.

1. $3x_1 - 2x_2 = 4$

   $x_1 - x_2 = -2$

2. $10x_1 + 15x_2 = 1$

   $3x_1 + 2x_2 = -1$

3. $x_1 + 2x_2 - x_3 = 0$

   $2x_1 + x_2 + 2x_3 = 9$

   $x_1 - x_2 + x_3 = 3$

4. $x_1 - x_2 - 2x_3 = 0$
   $2x_1 + 4x_2 + 5x_3 = 0$
   $6x_1 - 3x_3 = 0$

Find the rank of each of the below matrices.

5. $\begin{pmatrix} 2 & -2 \\ 0 & 0 \end{pmatrix}$

6. $\begin{pmatrix} 1 & 1 & 2 \\ -1 & 2 & 4 \\ -1 & 0 & 3 \end{pmatrix}$

7. $\begin{pmatrix} 3 & -2 & 2 & 0 \\ 6 & 2 & 4 & 5 \end{pmatrix}$

8. $\begin{pmatrix} 1 & -2 & 3 & 4 \\ 1 & 4 & 6 & 8 \\ 0 & 1 & 0 & 0 \\ 2 & 5 & 6 & 8 \end{pmatrix}$

Find the inverse of each of the below matrices, or show that one does not exist.

9. $\begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix}$

10. $\begin{pmatrix} 1 & 0 & -1 \\ 0 & -2 & 1 \\ 2 & -1 & 3 \end{pmatrix}$

11. $\begin{pmatrix} 8 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

Row reduce each of the following matrices to a triangular form. Then, use that to find the determinant of the original matrix.

12. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 5 \\ -3 & 0 & 2 \end{pmatrix}$

13. $\begin{pmatrix} 2 & 4 & 5 \\ 4 & 2 & 0 \\ 8 & 7 & -2 \end{pmatrix}$

14. $\begin{pmatrix} -2 & 2 & 6 \\ 5 & 0 & 1 \\ 1 & -2 & 2 \end{pmatrix}$

15. $\begin{pmatrix} 6 & 2 & 7 \\ -4 & -3 & 2 \\ 2 & 4 & 8 \end{pmatrix}$

Find all the eigenvalues and eigenvalues of the matrices below.

16. $\begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix}$

17. $\begin{pmatrix} 1 & 1 \\ \frac{1}{4} & 1 \end{pmatrix}$

18. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

19. $\begin{pmatrix} 7 & 0 \\ 0 & 13 \end{pmatrix}$

20. $\begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

21. $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

22. $\begin{pmatrix} 2 & -1 & 0 \\ 5 & 2 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

2.3 Conceptual Problems

1. Suppose $\det(A) = 1$. Then what is $\det(2A)$ if

   (a) A is a 2×2 matrix?
   (b) A is a 3×3 matrix?
   (c) A is a 4×4 matrix?
   (d) A is a 5×5 matrix?
   (e) A is a $n \times n$ matrix?

2. A permutation matrix is a matrix A each of whose columns and rows contain exactly one entry of 1, and all other entries 0. Some examples of permutation matrices are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

For a given permutation matrix A, what are all the possible values of $\det(A)$? Hint: use the properties of determinant under row reduction. You can try the examples above to get some ideas.

3. Without doing any computation, either find the inverse of the below matrix or argue why one does not exist:

$$\begin{pmatrix} 1 & 2 & 3 \\ 0 & \pi & .2 \\ 0 & 0 & 0 \end{pmatrix}$$

4. Without doing any computation, argue why the below system of equations does not have a unique solution:

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ x_1 - x_2 - x_3 &= 0 \end{aligned}$$

5. What is $\det(A)$ in each of the below cases?

   (a) The equation $AX = 0$ has a nontrivial solution.

   (b) $\lambda = 0$ is an eigenvalue of A.

   (c) A is singular.

   (d) A is a 3×3 matrix whose eigenvalues are 1, 2, and 3.

6. In this problem, we investigate orthogonal matrices:

   (a) Find the inverse of the matrix

$$\begin{pmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{pmatrix}$$

   (b) We say that a matrix A is orthogonal if $A^{-1} = A^T$. Verify that the matrix above is an orthogonal matrix.

   (c) Verify that the matrix below is an orthogonal matrix:

$$\begin{pmatrix} \frac{1}{\sqrt{3}} & 0 & \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

   (d) If A is an orthogonal matrix, what are the possibilities for $\det(A)$? Prove your answer. Hint: use properties of determinants and the fact that $AA^{-1} = I$.

7. Suppose I tell you that λ is an eigenvalue for the matrix A. What are the possibilities for the number of eigenvectors of A which correspond to λ? Select all that apply.

(a) 0
(b) 1
(c) 2
(d) infinitely many

8. Suppose A is an $n \times n$ matrix. Which of the following senarios are possible? Select all that apply.

   (a) A has less than n different eigenvalues.
   (b) A has exactly n different eigenvalues.
   (c) A has more than n different eigenvalues.

9. Suppose A is singular. Is it possible to find a matrix B so that AB is invertible? Why or why not?

A.7 Spring 2018 - Math 574 and 575

This section contains the following documents from my teaching assistant assignment for Math 574 and 575:

1. My recitation syllabus from Math 574
2. A solution set to an optional homework set for Math 574 at the end of the semester that was posted for the students to use for review
3. My recitation syllabus from Math 575
4. A diagnostic linear algebra quiz that was given to students early in the semester
5. A sample homework set solution that was given to the students and the homework grader

MATH 574 - Spring 2018 - Workshop Syllabus

Contact Information

Name: Matt Charnley
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Office Hours

Mondays from 2-4pm in Hill 606
or by appointment.

Course

This class is MATH 574 - Numerical Analysis II, lectured by Michael Vogelius. The course will cover numerical methods for solving linear systems of equations, matrix eigenvalues, minimization, and finite difference methods. See the instructor syllabus for more complete and accurate information.

Class Meetings

Lecture: W 1:40PM - 4:40PM, Hill 423
Workshop: Th 1:40PM - 3:00PM, ARC 203

Goals

The goals for these workshop sessions are to give you extra help and guidance in solving the types of problems covered in this course. While a lot of things will look familiar from 573, there are quite a few new ideas in this class that can be tricky to grasp at first. My role, and the role of these sessions, is to show you these ideas in more detail by working out problems so that you can figure out these concepts and understand how to use it in the future. I will also be able to provide support with any computer or programming issues you may have with the assignments.

Outline of Session

The general plan for each session is to begin by going over the material covered in lecture during the previous week. I will then take time to go over some problems from the previous week's homework that were particularly difficult. There will also be time at the end (as well as during the body of the sections) for questions from you, which can either be about material from lecture or problems on the upcoming homework set. In the lead-up to exams, these weekly sessions will turn into a review session for the exam.

Math 574 - Optional Homework Solutions

Matt Charnley

April 25, 2018

8. See the attached code. This computes the largest eigenvalue and corresponding eigenvector by the power method. This agrees with the information computed algebraically.

10. The same proof goes through as in the book. The only difference is that none of the terms x_i with eigenvalue λ_1 vanish. However, this is fine, because if x_1 and x_2 are both eigenvectors with eigenvalue λ_1, so is any linear combination of x_1 and x_2. Therefore, the most we can say is that the sequence of iterates converge to a linear combination of $x_1, x_2, ..., x_r$, but this is still an eigenvector with eigenvalue λ_1, which is the eigenvalue with largest magnitude.

11. Let $\{v_1, v_2, ..., v_n\}$ be an orthonormal basis of eigenvectors corresponding to eigenvalues $\lambda_1, ..., \lambda_n$. Then, for $x = \sum \alpha_i v_i$, we have

$$\mathcal{R}(x) = \frac{(Ax,x)}{(x,x)} = \frac{(\sum \alpha_i A v_i, \sum \alpha_j v_j)}{(\sum \alpha_i v_i, \sum \alpha_j v_j)} = \frac{\sum \lambda_i \alpha_i^2}{\sum \alpha_i^2}$$

Since $\lambda_1 \geq \lambda_i \geq \lambda_n$ for all i, we can see that

$$\lambda_1 = \frac{\lambda_1 \sum \alpha_i^2}{\sum \alpha_i^2} \geq \mathcal{R}(x) \geq \frac{\lambda_n \sum \alpha_i^2}{\sum \alpha_i^2} = \lambda_n$$

which clearly gives that $\max \mathcal{R}(x) \leq \lambda_1$ and $\min \mathcal{R}(x) \geq \lambda_n$. On the other hand, the same calculation as above gives that

$$\mathcal{R}(v_1) = \lambda_1 \qquad \mathcal{R}(v_n) = \lambda_n$$

Therefore, those inequalities are equalities, and we have

$$\max \mathcal{R}(x) = \lambda_1 \qquad \min \mathcal{R}(x) = \lambda_n$$

14. See the attached code. The functions go through the Householder matrices to find the QR factorization.

22. See the attached code. Looking at the end results, we can conclude that the eigenvalues for each matrix are:

(a) 3.7321, 2.0000, 0.2679

(b) 3.6180, 2.6180, 1.3820, 0.3820

(c) 2.9021, 2.1756, 1.0000, -0.9021, -0.1756

Optional Homework - MATH 574

Matt Charnley

Contents

Problem 8 - Power Method

```
% Part a)

clear all;

A = [[6,4,4,1];[4,6,1,4];[4,1,6,4];[1,4,4,6]];

[l1, x1] = PowerMethod(A);

disp(A);
disp(['Largest Eigenvalue: ' num2str(l1)]);
disp(['Eigenvector: [' num2str(x1') ']' ]);
disp(' ');
disp(' ');

% Part b)

clear all;

A = [[2,1,3,4];[1,-3,1,5];[3,1,6,-2];[4,5,-2,-1]];

[l1, x1] = PowerMethod(A);

disp(A);
disp(['Largest Eigenvalue: ' num2str(l1)]);
disp(['Eigenvector: [' num2str(x1') ']' ]);

disp(' ');
disp(' ');

% Part c)

clear all;

A = [[1,3,-2]; [-1,-2,3];[1,1,2]];

[l1, x1] = PowerMethod(A);

disp(A);
disp(['Largest Eigenvalue: ' num2str(l1)]);
disp(['Eigenvector: [' num2str(x1') ']' ]);

disp(' ');
disp(' ');
```

```
     6     4     4     1
     4     6     1     4
     4     1     6     4
     1     4     4     6

Largest Eigenvalue: 15
Eigenvector: [1           1           1           1]


     2     1     3     4
     1    -3     1     5
     3     1     6    -2
     4     5    -2    -1

Largest Eigenvalue: -8.0286
Eigenvector: [-0.38998    -0.97553      0.2955           1]


     1     3    -2
    -1    -2     3
     1     1     2

Largest Eigenvalue: 2.6289
Eigenvector: [-0.024463     0.65338           1]
```

Problem 14 - QR Factorization

```
A = [[1,1,1];[2,-1,-1];[2,-4,5]];

[Q,R] = QR_HH(A);
A
Q
R
Q*R

A = [[1,3,-2];[-1,-2,3];[1,1,2]];

[Q,R] = QR_HH(A);
A
Q
R
Q*R
```

```
A =

     1     1     1
     2    -1    -1
     2    -4     5


Q =

   -0.3333   -0.6667   -0.6667
   -0.6667   -0.3333    0.6667
   -0.6667    0.6667   -0.3333
```

```
R =

   -3.0000    3.0000   -3.0000
   -0.0000   -3.0000    3.0000
   -0.0000   -0.0000   -3.0000


ans =

    1.0000    1.0000    1.0000
    2.0000   -1.0000   -1.0000
    2.0000   -4.0000    5.0000


A =

     1     3    -2
    -1    -2     3
     1     1     2


Q =

   -0.5774   -0.7071    0.4082
    0.5774    0.0000    0.8165
   -0.5774    0.7071    0.4082


R =

   -1.7321   -3.4641    1.7321
         0   -1.4142    2.8284
         0         0    2.4495


ans =

    1.0000    3.0000   -2.0000
   -1.0000   -2.0000    3.0000
    1.0000    1.0000    2.0000
```

Problem 22

```
vals = [2,4,8,16,32, 64,128];

% Matrix (a) No Shift

A = [[3,1,0];[1,2,1];[0,1,1]];
disp('Matrix (a), No Shift');
disp(' ');

Ai = A;
for i=2:max(vals)
    [Q,R] = QR_HH(Ai);
    Ai = R*Q;
    if ismember(i,vals)
        disp(['After ' num2str(i) ' iterations:']);
```

```
        disp(Ai);
       disp(' ');
    end
end

% Matrix (a) with shift
disp(' ');
disp('Matrix (a), With Shift');
disp(' ');
Ai = A;
n = length(Ai);
for i=2:max(vals)
    a = Ai(n-1,n-1);
    b = Ai(n-1,n);
    c = Ai(n,n);
    l = a+c+sign(a)*sqrt((a+c)^2 + 4*(b^2 - a*c))/2;
    [Q,R] = QR_HH(Ai - l*eye(n));
    Ai = R*Q + l*eye(n);
    if ismember(i,vals)
        disp(['After ' num2str(i) ' iterations:']);
        disp(Ai);
        disp('');
    end
end
```

```
Matrix (a), No Shift

After 2 iterations:
    3.5000    0.5916    0.0000
    0.5916    2.2143   -0.1807
    0.0000   -0.1807    0.2857


After 4 iterations:
    3.7103    0.1927    0.0000
    0.1927    2.0217   -0.0031
    0.0000   -0.0031    0.2680


After 8 iterations:
    3.7319    0.0161    0.0000
    0.0161    2.0001   -0.0000
    0.0000   -0.0000    0.2679


After 16 iterations:
    3.7321    0.0001    0.0000
    0.0001    2.0000   -0.0000
   -0.0000   -0.0000    0.2679


After 32 iterations:
    3.7321    0.0000    0.0000
    0.0000    2.0000   -0.0000
   -0.0000   -0.0000    0.2679


After 64 iterations:
    3.7321   -0.0000    0.0000
```

```
    0.0000    2.0000   -0.0000
   -0.0000   -0.0000    0.2679


After 128 iterations:
    3.7321   -0.0000    0.0000
    0.0000    2.0000   -0.0000
   -0.0000   -0.0000    0.2679



Matrix (a), With Shift

After 2 iterations:
    1.5617    0.9023    0.0000
    0.9023    1.3650   -1.1455
    0.0000   -1.1455    3.0733

After 4 iterations:
    0.8769    0.8349   -0.0000
    0.8349    1.4451   -0.3322
   -0.0000   -0.3322    3.6780

After 8 iterations:
    0.3315    0.3257    0.0000
    0.3257    1.9373   -0.0386
   -0.0000   -0.0386    3.7312

After 16 iterations:
    0.2683    0.0261    0.0000
    0.0261    1.9996   -0.0009
    0.0000   -0.0009    3.7321

After 32 iterations:
    0.2679    0.0002    0.0000
    0.0002    2.0000   -0.0000
    0.0000   -0.0000    3.7321

After 64 iterations:
    0.2679    0.0000    0.0000
    0.0000    2.0000   -0.0000
    0.0000   -0.0000    3.7321

After 128 iterations:
    0.2679   -0.0000    0.0000
    0.0000    2.0000    0.0000
    0.0000   -0.0000    3.7321
```

```
% Matrix (b) No Shift

disp('Matrix (b), No Shift');
disp(' ');
A = [[2,1,0,0];[1,2,1,0];[0,1,2,1];[0,0,1,2]];

Ai = A;
for i=2:max(vals)
    [Q,R] = QR_HH(Ai);
    Ai = R*Q;
```

```
    if ismember(i,vals)
        disp(['After ' num2str(i) ' iterations:']);
        disp(Ai);
        disp(' ');
    end
end

% Matrix (a) with shift
disp(' ');
disp('Matrix (b), With Shift');
disp(' ');
Ai = A;
n = length(Ai);
for i=2:max(vals)
    a = Ai(n-1,n-1);
    b = Ai(n-1,n);
    c = Ai(n,n);
    l = a+c+sign(a+eps)*sqrt((a+c)^2 + 4*(b^2 - a*c))/2;
    [Q,R] = QR_HH(Ai - l*eye(n));
    Ai = R*Q + l*eye(n);
    if ismember(i,vals)
        disp(['After ' num2str(i) ' iterations:']);
        disp(Ai);
        disp(' ');
    end
end
```

```
Matrix (b), No Shift

After 2 iterations:
    2.8000    0.7483   -0.0000   -0.0000
    0.7483    2.3429    0.8748    0.0000
   -0.0000    0.8748    2.1905   -0.6236
   -0.0000   -0.0000   -0.6236    0.6667


After 4 iterations:
    3.3333    0.4689   -0.0000    0.0000
    0.4689    2.7585    0.4270    0.0000
   -0.0000    0.4270    1.5249   -0.0361
   -0.0000    0.0000   -0.0361    0.3832


After 8 iterations:
    3.5906    0.1635   -0.0000    0.0000
    0.1635    2.6447    0.0320    0.0000
   -0.0000    0.0320    1.3828   -0.0002
   -0.0000    0.0000   -0.0002    0.3820


After 16 iterations:
    3.6179    0.0126   -0.0000    0.0000
    0.0126    2.6182    0.0002    0.0000
   -0.0000    0.0002    1.3820   -0.0000
   -0.0000   -0.0000   -0.0000    0.3820


After 32 iterations:
    3.6180    0.0001   -0.0000    0.0000
```

```
    0.0001    2.6180    0.0000    0.0000
   -0.0000    0.0000    1.3820   -0.0000
   -0.0000   -0.0000   -0.0000    0.3820


After 64 iterations:
    3.6180    0.0000   -0.0000    0.0000
    0.0000    2.6180    0.0000    0.0000
    0.0000    0.0000    1.3820   -0.0000
    0.0000   -0.0000   -0.0000    0.3820


After 128 iterations:
    3.6180    0.0000   -0.0000    0.0000
    0.0000    2.6180    0.0000    0.0000
    0.0000    0.0000    1.3820   -0.0000
    0.0000   -0.0000   -0.0000    0.3820



Matrix (b), With Shift

After 2 iterations:
    1.4000    0.8602    0.0000   -0.0000
    0.8602    1.8703    0.9698         0
    0.0000    0.9698    1.9724   -0.9187
   -0.0000   -0.0000   -0.9187    2.7573


After 4 iterations:
    0.8939    0.6278    0.0000   -0.0000
    0.6278    1.5280    0.8038    0.0000
    0.0000    0.8038    2.1801   -0.4933
   -0.0000   -0.0000   -0.4933    3.3979


After 8 iterations:
    0.5352    0.3679    0.0000   -0.0000
    0.3679    1.3150    0.3284    0.0000
    0.0000    0.3284    2.5515   -0.1419
   -0.0000   -0.0000   -0.1419    3.5983


After 16 iterations:
    0.3931    0.1049   -0.0000   -0.0000
    0.1049    1.3721    0.0393    0.0000
   -0.0000    0.0393    2.6170   -0.0150
   -0.0000   -0.0000   -0.0150    3.6178


After 32 iterations:
    0.3820    0.0068   -0.0000   -0.0000
    0.0068    1.3819    0.0006    0.0000
    0.0000    0.0006    2.6180   -0.0002
   -0.0000   -0.0000   -0.0002    3.6180


After 64 iterations:
    0.3820    0.0000   -0.0000   -0.0000
    0.0000    1.3820    0.0000    0.0000
   -0.0000    0.0000    2.6180   -0.0000
```

```
   -0.0000   -0.0000   -0.0000    3.6180


After 128 iterations:
    0.3820    0.0000   -0.0000   -0.0000
    0.0000    1.3820    0.0000    0.0000
   -0.0000    0.0000    2.6180   -0.0000
    0.0000   -0.0000   -0.0000    3.6180
```

```
% Matrix (c) No Shift

A = [[0,1,0,0,0];[1,1,1,0,0];[0,1,1,1,0];[0,0,1,1,1];[0,0,0,1,2]];

disp('Matrix (c), No Shift');
disp(' ');

Ai = A;
for i=2:max(vals)
    [Q,R] = QR_HH(Ai);
    Ai = R*Q;
    if ismember(i,vals)
        disp(['After ' num2str(i) ' iterations:']);
        disp(Ai);
        disp(' ');
    end
end

% Matrix (c) with shift
disp(' ');
disp('Matrix (c), With Shift');
disp(' ');
Ai = A;
n = length(Ai);
for i=2:max(vals)
    a = Ai(n-1,n-1);
    b = Ai(n-1,n);
    c = Ai(n,n);
    l = a+c+sign(a)*sqrt((a+c)^2 + 4*(b^2 - a*c))/2;
    [Q,R] = QR_HH(Ai - l*eye(n));
    Ai = R*Q + l*eye(n);
    if ismember(i,vals)
        disp(['After ' num2str(i) ' iterations:']);
        disp(Ai);
        disp(' ');
    end
end
```

```
Matrix (c), No Shift

After 2 iterations:
    1.0000   -1.4142         0         0         0
   -1.4142    0.5000   -0.8660   -0.0000    0.0000
   -0.0000   -0.8660    1.5000    0.8165    0.0000
   -0.0000   -0.0000    0.8165    2.0000    0.5774
    0.0000    0.0000    0.0000    0.5774    0.0000
```

```
After 4 iterations:
    2.4000   -0.5538    0.0000   -0.0000   -0.0000
   -0.5538    0.7739   -1.6547    0.0000    0.0000
    0.0000   -1.6547    0.9596    0.5072   -0.0000
   -0.0000   -0.0000    0.5072    1.0420    0.0105
   -0.0000   -0.0000    0.0000    0.0105   -0.1755


After 8 iterations:
    2.8078   -0.2443   -0.0000   -0.0000   -0.0000
   -0.2443    2.2675   -0.0822    0.0000    0.0000
   -0.0000   -0.0822   -0.7605    0.4959   -0.0000
    0.0000    0.0000    0.4959    0.8608    0.0000
    0.0000    0.0000   -0.0000    0.0000   -0.1756


After 16 iterations:
    2.9010   -0.0279   -0.0000   -0.0000   -0.0000
   -0.0279    2.1766   -0.0001    0.0000   -0.0000
    0.0000   -0.0001   -0.3480    0.8643   -0.0000
   -0.0000   -0.0000    0.8643    0.4459    0.0000
   -0.0000   -0.0000   -0.0000    0.0000   -0.1756


After 32 iterations:
    2.9021   -0.0003   -0.0000   -0.0000   -0.0000
   -0.0003    2.1756   -0.0000    0.0000   -0.0000
   -0.0000   -0.0000    0.8429    0.5236   -0.0000
   -0.0000   -0.0000    0.5236   -0.7450   -0.0000
   -0.0000   -0.0000   -0.0000    0.0000   -0.1756


After 64 iterations:
    2.9021   -0.0000   -0.0000    0.0000   -0.0000
   -0.0000    2.1756    0.0000    0.0000   -0.0000
    0.0000   -0.0000    0.9998    0.0211   -0.0000
    0.0000   -0.0000    0.0211   -0.9019   -0.0000
    0.0000   -0.0000    0.0000    0.0000   -0.1756


After 128 iterations:
    2.9021   -0.0000   -0.0000    0.0000   -0.0000
   -0.0000    2.1756    0.0000    0.0000   -0.0000
    0.0000   -0.0000    1.0000    0.0000   -0.0000
    0.0000   -0.0000    0.0000   -0.9021   -0.0000
    0.0000   -0.0000    0.0000    0.0000   -0.1756



Matrix (c), With Shift

After 2 iterations:
   -0.4029    0.7003    0.0000    0.0000   -0.0000
    0.7003    0.7406    0.9413    0.0000   -0.0000
         0    0.9413    0.9480    0.9887    0.0000
         0    0.0000    0.9887    1.1215   -0.5925
         0   -0.0000   -0.0000   -0.5925    2.5928


After 4 iterations:
```

```
   -0.6638    0.4328   -0.0000    0.0000    0.0000
    0.4328    0.2651    0.7533    0.0000   -0.0000
    0.0000    0.7533    0.7815    0.7219    0.0000
    0.0000   -0.0000    0.7219    1.7830   -0.2397
   -0.0000    0.0000   -0.0000   -0.2397    2.8341


After 8 iterations:
   -0.8162    0.2434   -0.0000   -0.0000    0.0000
    0.2434   -0.1310    0.3816    0.0000   -0.0000
   -0.0000    0.3816    0.9063    0.2109    0.0000
   -0.0000    0.0000    0.2109    2.1466   -0.0756
    0.0000   -0.0000    0.0000   -0.0756    2.8943


After 16 iterations:
   -0.8891    0.0965   -0.0000   -0.0000    0.0000
    0.0965   -0.1855    0.0608    0.0000   -0.0000
    0.0000    0.0608    0.9971    0.0174    0.0000
    0.0000    0.0000    0.0174    2.1755   -0.0100
   -0.0000   -0.0000    0.0000   -0.0100    2.9020


After 32 iterations:
   -0.9018    0.0140   -0.0000   -0.0000    0.0000
    0.0140   -0.1758    0.0014    0.0000   -0.0000
    0.0000    0.0014    1.0000    0.0001    0.0000
    0.0000    0.0000    0.0001    2.1756   -0.0002
   -0.0000   -0.0000    0.0000   -0.0002    2.9021


After 64 iterations:
   -0.9021    0.0003   -0.0000   -0.0000    0.0000
    0.0003   -0.1756    0.0000    0.0000   -0.0000
    0.0000    0.0000    1.0000    0.0000    0.0000
    0.0000    0.0000    0.0000    2.1756   -0.0000
   -0.0000   -0.0000    0.0000   -0.0000    2.9021


After 128 iterations:
   -0.9021    0.0000   -0.0000   -0.0000    0.0000
    0.0000   -0.1756    0.0000    0.0000   -0.0000
   -0.0000    0.0000    1.0000   -0.0000    0.0000
   -0.0000    0.0000    0.0000    2.1756   -0.0000
    0.0000   -0.0000    0.0000   -0.0000    2.9021
```

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```
function [ lam1, x1, Rvec, lit ] = PowerMethod(A)
%PowerMethod - Uses the Power Method to compute the largest eigenvalue of
%the matrix A

n = length(A);
x0 = rand(n,1);
Rvec= [1,1,1];
step = 1;
tol = 10^(-7);
k = floor(n/2);
z = x0;
lm2 = 0;
lm1 = 0;
lit = [];
while step < 5 || abs(Rvec(step) - Rvec(step-1)) > tol
    w = A*z;
    while z(k) < tol
        k = k+1;
        if k == n+1
            k=1;
        end
    end
    lm = w(k)/z(k);
    lit(end+1) = lm;
    if step > 2
        if abs(lm1 - lm2) < tol^2;
            Rvec(step+1) = Rvec(step);
        else
            Rvec(step+1) = (lm - lm1)/(lm1 - lm2);
        end
    end
    lm2 = lm1;
    lm1 = lm;
    z = w/max(abs(w));
    step  = step + 1;
end

lam1 = lm;
x1 = z;


end
```

```
Error using PowerMethod (line 5)
Not enough input arguments.
```

Published with MATLAB® R2015a

```
function [ Q,R ] = QR_HH( A )
%QR_HH - does the QR factorization of a matrix A using Householder matrices
%   Detailed explanation goes here

n = length(A);
Q = eye(n);
R = A;

for i=1:n-1
    b = R(:,i);
    w = HH_w(b, i);
    P = eye(n) - 2*w*w';
    R = P*R;
    Q = Q*P.';
end

end
```

```
Error using QR_HH (line 5)
Not enough input arguments.
```

Published with MATLAB® R2015a

```
function [ w ] = HH_w( b, r )
%HH_w - returns the vector w so that for U = I - 2ww^*, we have Ub with the
%last r+1 entries zero.
%   Detailed explanation goes here
n = length(b);
w = zeros(n,1);
if r <= n
   d = b(r:n);
   v = zeros(length(d),1);
   alpha = -norm(d, 2)*sign(d(1)+eps);
   v(1) = sqrt(1/2 * (1 - d(1)/alpha));
   p = -alpha*v(1);
   v(2:end) = d(2:end)/(2*p);
   w(r:n) = v;
end

end
```

```
Error using HH_w (line 5)
Not enough input arguments.
```

Published with MATLAB® R2015a

MATH 575 - Spring 2018 - Workshop Syllabus

Contact Information

Name: Matt Charnley
Email: charnley@math.rutgers.edu
Office: Hill 606

Office Hours

Mondays from 2-4pm in Hill 606
or by appointment.

Course

This class is MATH 575 - Numerical Solution of Partial Differential Equations, lectured by Paul Feehan. The course will cover both finite element and finite difference methods for elliptic, parabolic, and hyperbolic equations. See the instructor's syllabus for more complete and accurate information.

Class Meetings

Lecture: TF 12:00PM - 1:20PM, CORE 101
Workshop: Th 10:20AM - 11:40AM, Hill 005

Goals

The goals for these workshop sessions are to give you extra help and guidance in solving the types of problems covered in this course. While a lot of things will look familiar from 573, there are quite a few new ideas in this class that can be tricky to grasp at first. My role, and the role of these sessions, is to show you these ideas in more detail by working out problems so that you can figure out these concepts and understand how to use it in the future. I will also be able to provide support with any computer or programming issues you may have with the assignments.

Outline of Session

The general plan for each session is to begin by going over the material covered in lecture during the previous week. I will then take time to go over some problems from the previous week's homework that were particularly difficult. There will also be time at the end (as well as during the body of the sections) for questions from you, which can either be about material from lecture or problems on the upcoming homework set. In the lead-up to exams, these weekly sessions will turn into a review session for the exam.

MATH 575 - Linear Algebra Diagnostic Quiz

June 6, 2018

Name: ____

1. Let A be an $n \times n$ matrix over $\mathbb{R}$. What does it mean for A to be
   (a) symmetric?
   (b) orthogonal?

2. Let A be an $n \times n$ matrix over $\mathbb{C}$. What does it mean for A to be
   (a) Hermitian?
   (b) unitary?

3. There are several properties of a matrix A that are equivalent to A being invertible. State the three of these related to
   (i) determinants,
   (ii) eigenvalues, and
   (iii) solutions to $Ax = 0$.

4. Let A, B, and P be matrices of appropriate sizes. For each computation below, relate it to related computations for the individual matrices, or say that there's no way to simplify it.
   (a) $\mathrm{tr}(AB)$
   (b) $\mathrm{tr}(P^{-1}AP)$
   (c) $\det(AB)$
   (d) $\det(P^{-1}AP)$

5. What does it mean for a real matrix A to be positive definite? You should have two definitions here, one involving eigenvalues, and one involving arbitrary vectors in $\mathbb{R}^n$.

6. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, ..., \lambda_n$, possibly repeated. What are $\mathrm{tr}(A)$ and $\det(A)$?

7. Let A be an $n \times n$ real matrix and assume that there is a matrix B so that

$$\langle Ax, y \rangle = \langle x, By \rangle$$

for all vectors x and y, where this is a real inner product. What relation must A and B satisfy?

8. Let A be an $n \times n$ complex matrix and assume that there is a matrix B so that

$$\langle Ax, y \rangle = \langle x, By \rangle$$

for all vectors x and y, where this is a complex inner product. What relation must A and B satisfy?

9. What conditions will guarantee that a matrix A has real eigenvalues?

10. What conditions will guarantee that a matrix A is diagonalizable?

11. Show that if A is a real matrix and λ is an eigenvalue of A, $\bar{\lambda}$ is also an eigenvalue of A.

12. If a matrix is not diagonalizable, what is the best replacement for diagonalizing A? What does this matrix look like?

Math 575 - Homework 9 Solutions

Matt Charnley

April 19, 2018

1. Using the fact that the vertices are $\mathbf{a}_1 = (1,0)$, $\mathbf{a}_2 = (1,1)$, and $\mathbf{a}_3 = (0,1)$, we can see that the function $1-x_1$ is zero on both $\mathbf{a}_1$ and $\mathbf{a}_2$ and is 1 at $\mathbf{a}_3$. Therefore $\lambda_3 = 1-x_1$. By a similar argument, $\lambda_1 = 1-x_2$, and then by the fact that all of the all of the barycentric coordinates need to add to 1, we have $\lambda_2 = x_1+x_2-1$.

2.

(a) Since $\mathbf{m}_{ij}$ is the point where $\lambda_i = \lambda_j = \frac{1}{2}$, we get that if ϕ_k is of the form $c_1\lambda_1 + c_2\lambda_2 + c_3\lambda_3$, then it must satisfy

$$1 = \phi_k(\mathbf{m}_{ij}) = \frac{1}{2}(c_i + c_j) \qquad 0 = \phi_k(\mathbf{m}_{ik}) = \frac{1}{2}(c_i + c_k) \qquad 0 = \phi_k(\mathbf{m}_{jk}) = \frac{1}{2}(c_k + c_j)$$

Solving this system gives $c_i = c_j = 1$ and $c_k = -1$. Thus, we have that

$$\phi_k = \lambda_i + \lambda_j - \lambda_k = 1 - 2\lambda_k$$

because the sum of all of the lambdas is 1. Therefore

$$\phi_1 = 1 - 2\lambda_1 \qquad \phi_2 = 1 - 2\lambda_2 \qquad \phi_3 = 1 - 2\lambda_3$$

(b) Therefore, if we want to find a polynomial on T so that $P(\mathbf{m}_{12}) = 4$, $P(\mathbf{m}_{23}) = 5$, $P(\mathbf{m}_{31}) = 6$, then we can just do this by

$$P = 4\phi_3 + 5\phi_1 + 6\phi_2$$

3.

(a) Let P be a quadratic polynomial on a triangle T with value zero at all the vertices, and average value zero on all of the edges. Then, since the quadrature formula is exact, we know that

$$0 = \int_{e_{ij}} P(t)\ dt = C[P(\mathbf{a}_i) + 4P(\mathbf{m}_{ij}) + P(\mathbf{a}_j)] = 4CP(\mathbf{m}_{ij})$$

because the value at the vertices are zero. Thus, we have that $P(\mathbf{m}_{ij}) = 0$ as well for each edge. Therefore, from the representation formula

$$P(x) = \sum_{i=1}^{3} \lambda_i(2\lambda_i - 1)P(a_i) + \sum_{i<j} 4\lambda_i\lambda_j P(m_{ij})$$

since the value at each vertex and midpoint is zero, we have that $P \equiv 0$.

(b) If g is a quadratic polynomial, then for both of the integrals

$$\int_a^b g(t)\ dt \qquad \text{and} \qquad \int_a^b g(t)(t - [a+b]/2)\ dt$$

the quadrature formula exactly holds. Thus, we have that

$$0 = \frac{b-a}{6}[g(a) + 4g([a+b]/2) + g(b)]$$
$$0 = \frac{b-a}{6}[g(a)(a - [a+b]/2) + 4g([a+b]/2)[a+b]/2 - [a+b]/2) + g(b)(b - [a+b]/2)]$$

Simplifying these equations, we get

$$0 = g(a) + 4g([a+b]/2) + g(b) \qquad 0 = g(a)\frac{a-b}{2} + g(b)\frac{b-a}{2}$$

The second of these implies that $g(a) = g(b)$, and plugging this into the first equation gives that $g([a+b]/2) = -\frac{1}{2}g(a)$, as desired.

(c) Based on the motivation from the previous parts, we want to define a polynomial so that

$$g(a_i) = 2 \qquad g(m_{ij}) = -1$$

for all vertices a_i and midpoints m_{ij}. Thus, the polynomial

$$P(x) = \sum_{i=1}^{3} 2\lambda_i(2\lambda_i - 1) - 4\sum_{i<j} \lambda_i \lambda j$$

will be non-zero, but will satisfy

$$\int_e P\ ds = 0 \qquad \int_e Ps\ ds = 0$$

because these can be reduced to integrals of the form

$$\int_a^b g(t)\ dt \qquad \text{and} \int g(t)t\ dt$$

which are zero by the way we chose our endpoints and midpoints.

MATH 575 - Quiz 10

April 19, 2018

Name: Key.

1. Write out a pseudocode procedure for the Solve-Estimate-Mark-Refine method for adaptive mesh refinement. Assume that the method will stop then the total error is less than *tol*.

Start with an initial mesh T_0.

$k=0$.

SOLVE: Find the solution u_k on the mesh T_k.

ESTIMATE: Compute the total error and error on each triangle.

If the total error is $\leq tol$, then stop.

MARK: Pick a minimal set of triangles $M_k \subset T_k$ so that the total error on M_k is at least $\theta \cdot$ the error on the entire mesh, for a chosen $\theta \in (0,1)$.

REFINE: Refine the triangles in M_k to get the next mesh T_{k+1}.

Increment k.

2. Prove that the set

$$K := \{v \in H_0^1(\Omega)\ v \geq \psi \text{ a.e. in } \Omega\}$$

is a convex subset of $H_0^1(\Omega)$.

For any $u, v \in K$ and any $t \in (0,1)$, we know that

$$tu + (1-t)v \in H_0^1(\Omega) \quad \text{and}$$

$$tu + (1-t)v \geq t\psi + (1-t)\psi = \psi \quad \text{a.e. in } \Omega$$

Therefore $tu + (1-t)v \in K$, and so K is convex.

3. Consider the set K from the previous page,

$$K := \{v \in H_0^1(\Omega)\ v \geq \psi \text{ a.e. }\}$$

and define $u \in K$ by the solution to the variational inequality

$$\int_\Omega (\nabla u) \cdot \nabla(v-u)\,dx \geq \int_\Omega f(v-u)\,dx \qquad \forall v \in K$$

Prove that:

(a) $-\Delta u \geq f$ a.e. in Ω. (Hint: Integrate by parts and choose v appropriately)

Integration gives $\int_\Omega -\Delta u\,(v-u) \geq \int_\Omega f\,(v-u)$

For any $w \in H_0^1(\Omega)$, with $w \geq 0$, we can pick $v = u + w$ so that this reads $\int_\Omega (-\Delta u) w \geq \int_\Omega f w \quad \forall w \in H_0^1(\Omega)\ w \geq 0$

Thus $-\Delta u \geq f$. ✓.

(b) $u \geq \psi$ a.e. in Ω.

This is in the definition of K.

(c) $(-\Delta u - f)(u - \psi) = 0$ a.e. in Ω (Hint: Assume $u(x) > \psi(x)$. I can then pick a function $v \in K$ so that $v \leq u$, and strictly less near x. What does this do to the inequality?)

If $u(x) > \psi(x)$, $\exists$ a function $\eta(x) \geq 0$, compact support, $\eta(x) > 0$, so that $u(x) \pm \eta(x) > \psi$. Since η is positive and smooth with compact support, we also know that $u + \eta \in K$, $u - \eta \in K$.

Thus, applying the variational inequality to both $v = u + \eta$ and $v = u - \eta$, we get

$$\int -\Delta u(\eta) \geq \int f\eta \qquad \text{and} \qquad \int -\Delta u(-\eta) \geq \int f(-\eta).$$

which implies that $\int_\Omega -\Delta u\,\eta = \int_\Omega f\eta$

Therefore $-\Delta u = f$ on the support of η, i.e. where $u > \psi$.

Appendix B

Full Student Feedback

Included are the full SIRS student feedback surveys for each of the classes for which I was a teaching assistant. They have each been summarized in their corresponding sections, but the full results are shown here for reference. The classes are presented in the following order:

1. Fall 2014 - Math 135
2. Spring 2015 - Math 251
3. Fall 2015 - Math 251
4. Fall 2017 - Math 421
5. Spring 2018 - Math 575

Rutgers University Student Instructional Rating
(Online Survey)

Charnley M Fall 2014, 01:640:135:70 — Calculus I (index #14802) Enrollment= 30, Responses= 17	Student Responses						Weighted Means			
Part A: University-wide Questions:	Strong Disagree 1				Strong Agree 5	No response	Section	Course	Level	Dept
1. The instructor was prepared for class and presented the material in an organized manner	0	0	0	3	14	0	**4.82**	4.14	4.26	**4.29**
2. The instructor responded effectively to student comments and questions	0	0	0	1	16	0	**4.94**	3.97	4.14	**4.17**
3. The instructor generated interest in the course material	0	0	2	2	13	0	**4.65**	3.79	3.99	**4.01**
4. The instructor had a positive attitude toward assisting all students in understanding course material	0	0	0	1	16	0	**4.94**	4.17	4.30	**4.32**
5. The instructor assigned grades fairly	0	0	0	2	15	0	**4.88**	4.11	4.19	**4.21**
6. The instructional methods encouraged student learning	0	0	1	1	15	0	**4.82**	3.74	3.93	**3.96**
7. I learned a great deal in this course	0	0	0	4	13	0	**4.76**	3.68	3.89	**3.96**
8. I had a strong prior interest in the subject matter and wanted to take this course	0	1	3	2	11	0	**4.35**	3.34	3.45	**3.53**
	Poor				Excellent					
9. I rate the teaching effectiveness of the instructor as	0	0	0	4	13	0	**4.76**	3.74	3.93	**3.98**
10. I rate the overall quality of the course as	0	0	0	5	12	0	**4.71**	3.68	3.82	**3.87**

What do you like best about this course?:

"Teacher is very friendly; he closes the distance by memorizing every single students' name after one or two classes albeit having a total of 7,80+ students - impressive. When asking for help in office hours, he provides much assistance and simplifies explanations in a good way."

"The small size of the recitation allowed for many questions to be answered personally. "

"It helped review subjects in calculus class that might have otherwise been left confusing to understand"

"Matt did a great job of answering our questions, and helping us through the hard concepts. He was very relatable. "

"Mr. Charnley knows a lot about calc and it really helped me understand the material better when he went over it in recitation."

"The clarity in explaining the information."

"I liked my instructor's teaching style and the way he thoroughly taught us what we needed to know before the quiz. I also appreciated his willingness to help us by having office hours and helping me over email."

If you were teaching this course, what would you do differently?:

"Possibly check what the homework problems were to better understand the confusion that students have."

"Nothing, I thought he did a good job."

"Nothing"

"Keep it the same!"

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

"It helped as a sort of review on subjects and give the same lesson from a different teacher."

"Attitude towards class is very positive and instructor was always in a good mood"

"He has helped me understand calculus better."

"None"

"He helped my intellectual growth and progress by teaching me different methods to do problems, and encouraging us to ask questions. "

Other comments or suggestions::

"None"

"Thank you for everything! "

Rutgers University Student Instructional Rating
(Online Survey)

Charnley M Fall 2014, 01:640:135:71 — Calculus I (index #14804) Enrollment= 29, Responses= 15 **Part A: University-wide Questions:**	**Student Responses** **Strong Disagree 1**				**Strong Agree 5**	**No response**	**Weighted Means** **Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner	0	0	1	5	9	0	**4.53**	4.14	4.26	**4.29**
2. The instructor responded effectively to student comments and questions	0	0	1	3	11	0	**4.67**	3.97	4.14	**4.17**
3. The instructor generated interest in the course material	0	0	3	2	9	1	**4.43**	3.79	3.99	**4.01**
4. The instructor had a positive attitude toward assisting all students in understanding course material	0	0	1	2	12	0	**4.73**	4.17	4.30	**4.32**
5. The instructor assigned grades fairly	0	0	2	2	11	0	**4.60**	4.11	4.19	**4.21**
6. The instructional methods encouraged student learning	0	0	1	4	10	0	**4.60**	3.74	3.93	**3.96**
7. I learned a great deal in this course	0	1	1	4	9	0	**4.40**	3.68	3.89	**3.96**
8. I had a strong prior interest in the subject matter and wanted to take this course	3	1	2	3	6	0	**3.53**	3.34	3.45	**3.53**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as	0	0	1	5	9	0	**4.53**	3.74	3.93	**3.98**
10. I rate the overall quality of the course as	0	0	2	4	9	0	**4.47**	3.68	3.82	**3.87**

What do you like best about this course?:

"he was very effective in helping everyone who didn't understand the information in lecture understand it through his class"

" TA is really helpful!"

"He goes over things quickly"

"Definitely not my grades but I do like finding things out and linking concepts. The TA is helpful and supportive of that, even when its not during his office hours I have emailed him and received a quick response with help but not a totally detailed solution-- he helps you solve the problem."

"The effective teaching. "

"The reviewing of the material learned in class"

"practice is helpful"

"It was a great supplement to the regular lectures. Being able to go over various questions and topics that maybe were not covered to the desired depth in lecture was greatly helpful."

"I improved on simple algebra… "

"That it was very easy"

"helped me understand further understand materials I learned in lecture"

If you were teaching this course, what would you do differently?:

"nothing"

"Maybe use better examples or relate more to the professor"

"I would immediately quit because I suck at calculus. But seriously I would probably not do anything differently. The TA is great."

"Nothing. "

"nothing"

"more practice and review"

"Nothing comes to mind."

"I really don't know considering the fact that I can hardly do math, so probably nothing."

"Probably make webwork more user friebdky"

"nothing"

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

"made me understand the content even more"

"I mean it taught me math. So. There's that."

"He just helped me understand the material. It isn't a easy subject for me to understand, but he made it easy for me to understand. "

"He helped clear up any of the material that was confusing to me in lecture" “practice and review”

“He worked to make sure that every student's questions were answered to their satisfaction. He helped me achieve a deeper understanding of calculus, a topic I expected to struggle with.”

“Every time I failed a quiz, it would encourage to try to do better. ”

“He taught in a very easy to understand manner”

“I know now how to study for math classes”

Other comments or suggestions::

“The TA is really nice and chill. Give him a raise or something lol”

“I would definitely recommend this TA to other students. ”

“more classes, it's too short ”

“Calculus is hard, I liked the TA, it was the subject that was difficult for me. ”

“Hes a very good TA, very helpful”

“Great job Charnley”

Rutgers University Student Instructional Rating
(Online Survey)

Charnley M Fall 2014, 01:640:135:72 — Calculus I (index #14805) Enrollment= 31, Responses= 11 **Part A: University-wide Questions:**	**Student Responses** Strong Disagree 1				Strong Agree 5	No response	**Weighted Means** Section	Course	Level	Dept
1. The instructor was prepared for class and presented the material in an organized manner	0	0	0	3	7	1	**4.70**	4.14	4.26	**4.29**
2. The instructor responded effectively to student comments and questions	0	0	1	2	7	1	**4.60**	3.97	4.14	**4.17**
3. The instructor generated interest in the course material	0	0	1	1	7	2	**4.67**	3.79	3.99	**4.01**
4. The instructor had a positive attitude toward assisting all students in understanding course material	0	0	0	1	9	1	**4.90**	4.17	4.30	**4.32**
5. The instructor assigned grades fairly	0	0	0	2	8	1	**4.80**	4.11	4.19	**4.21**
6. The instructional methods encouraged student learning	0	0	2	1	7	1	**4.50**	3.74	3.93	**3.96**
7. I learned a great deal in this course	0	1	0	3	6	1	**4.40**	3.68	3.89	**3.96**
8. I had a strong prior interest in the subject matter and wanted to take this course	0	3	2	2	3	1	**3.50**	3.34	3.45	**3.53**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as	0	0	1	3	6	1	**4.50**	3.74	3.93	**3.98**
10. I rate the overall quality of the course as	0	0	1	4	5	1	**4.40**	3.68	3.82	**3.87**

What do you like best about this course?:

“I hate calculus, but the TA did his best in his teachings. I appreciated that.”

“The instructor was very prepared and taught in a way which assisted in the understanding of the concepts taught in lecture.”

“His teaching style”

“He went over the lectures in summary and answered specific questions”

“He was extremely helpful when anybody had questions.”

If you were teaching this course, what would you do differently?:

“Nothing.”

“I honestly wouldn't do anything differently; for an instructor teaching for the first time, he did an excellent job and his teaching methods are fine as they are.”

“Provide more explanation”

“Nothing”

“There is nothing more to do differently.”

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

“None.”

“I learned new ways to solve problems and it reinforced my understanding of the course material that was taught in lecture.”

“Helped me understand the concepts faster”

“Michael made a calculus easier to understand. He was very approachable and I didn't ever feel hesitant asking questions in class. I think you would make a great professor one day because he's very enthusiastic about this class and clearly cares a lot about his students. The one thing I would encourage him to do is to speak a bit more clearly when explaining the steps to a problem. Other than that I thought Michael was an excellent recitation instructor”

“Instructor answered any questions asked”

“He really helped me whenever i had a problem, thus he helped me improve on difficult questions. ”

Other comments or suggestions::

“None.”

“None.”

“He was very fun and the class was enjoyable. ”

Rutgers University Student Instructional Rating
(Online Survey)

Charnley Matthew Spring 2015, 01:640:251:22 — Multivariable Calc (index #19841) Enrollment= 26, Responses= 12 **Part A: University-wide Questions:**	Student Responses Strong Disagree 1				Strong Agree 5	No response	Weighted Means Section	Course	Level	Dept
1. The instructor was prepared for class and presented the material in an organized manner	1	0	2	0	9	0	**4.33**	4.47	4.46	**4.37**
2. The instructor responded effectively to student comments and questions	1	0	1	1	9	0	**4.42**	4.42	4.36	**4.29**
3. The instructor generated interest in the course material	1	0	2	1	8	0	**4.25**	4.24	4.16	**4.13**
4. The instructor had a positive attitude toward assisting all students in understanding course material	1	0	1	1	9	0	**4.42**	4.54	4.47	**4.41**
5. The instructor assigned grades fairly	1	0	0	3	8	0	**4.42**	4.32	4.35	**4.24**
6. The instructional methods encouraged student learning	1	0	1	3	7	0	**4.25**	4.16	4.13	**4.08**
7. I learned a great deal in this course	1	0	1	2	8	0	**4.33**	4.26	4.24	**4.10**
8. I had a strong prior interest in the subject matter and wanted to take this course	1	0	3	1	7	0	**4.08**	4.03	3.94	**3.62**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as	2	0	0	1	9	0	**4.25**	4.27	4.20	**4.11**
10. I rate the overall quality of the course as	1	1	1	1	8	0	**4.17**	4.14	4.14	**3.99**

What do you like best about this course?:

"Matt has a very upbeat personality and really seems like he enjoy teaching."

"The TA."

"Professor Charnley was very helpful"

"I liked the way he taught us it was very easy and i didnt had any difficulty understanding him "

If you were teaching this course, what would you do differently?:

"Alternate quizzes and maple labs every week. It was like that for a while but then we would have a quiz and lab due the same day."

"Nothing."

"Nothing"

"I would do reviews before every exams "

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

"Eh."

"It has allowd me to study harder "

Other comments or suggestions::

"Went to office hours ever change I got, learned everything there and nothing in lecture"

"stop being condescending "

"Overall this is a very nice class "

Rutgers University Student Instructional Rating
(Online Survey)

Charnley Matthew Spring 2015, 01:640:251:23 — Multivariable Calc (index #19842) Enrollment= 28, Responses= 8	**Student Responses**						**Weighted Means**			
Part A: University-wide Questions:	**Strong Disagree 1**				**Strong Agree 5**	**No response**	**Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner	0	0	0	0	8	0	**5.00**	4.47	4.46	**4.37**
2. The instructor responded effectively to student comments and questions	0	0	0	0	8	0	**5.00**	4.42	4.36	**4.29**
3. The instructor generated interest in the course material	0	0	1	2	5	0	**4.50**	4.24	4.16	**4.13**
4. The instructor had a positive attitude toward assisting all students in understanding course material	0	0	0	0	8	0	**5.00**	4.54	4.47	**4.41**
5. The instructor assigned grades fairly	0	1	0	1	6	0	**4.50**	4.32	4.35	**4.24**
6. The instructional methods encouraged student learning	0	0	0	2	6	0	**4.75**	4.16	4.13	**4.08**
7. I learned a great deal in this course	0	0	1	2	5	0	**4.50**	4.26	4.24	**4.10**
8. I had a strong prior interest in the subject matter and wanted to take this course	0	0	1	1	6	0	**4.63**	4.03	3.94	**3.62**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as	0	0	0	1	7	0	**4.88**	4.27	4.20	**4.11**
10. I rate the overall quality of the course as	0	0	1	1	6	0	**4.63**	4.14	4.14	**3.99**

What do you like best about this course?:

"Charnleye xplanation"

"I liked that it was not an overbearing workload, but it still taught me alot."

"Everything"

"the recitations summarized what was gone over in lecture and presented the material in a different, more to the point way. "

"I'll love math to my death, and this TA did everything to make sure the students understood"

If you were teaching this course, what would you do differently?:

"Na"

"The recitations were a perfect supplement to the lectures, which were sometimes very confusing. Matt usually had a much clearer and organized way of teaching and explaining things."

"Nothing, the instructor does a great job at making recitation what it's supposed to be."

"Nothing"

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

"explanined many diffilcut concepts intuitiveyl in reciation"

"Taught me how to integrate e^x^2 which was really cool"

"Further encouraged my mathematics major"

Other comments or suggestions::

"Na"

"Great teaching style and super helpful."

"Great guy and great instructor!"

"very helpful, a caring TA and enthusiastic about the material."

"Nothing"

Rutgers University Student Instructional Rating
(Online Survey)

Charnley Matthew Spring 2015, 01:640:251:24 — Multivariable Calc (index #19843) Enrollment= 24, Responses= 9	**Student Responses**						**Weighted Means**			
Part A: University-wide Questions:	**Strong Disagree 1**				**Strong Agree 5**	**No response**	**Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner	0	0	0	0	9	0	**5.00**	4.47	4.46	**4.37**
2. The instructor responded effectively to student comments and questions	0	0	0	0	9	0	**5.00**	4.42	4.36	**4.29**
3. The instructor generated interest in the course material	0	0	1	1	7	0	**4.67**	4.24	4.16	**4.13**
4. The instructor had a positive attitude toward assisting all students in understanding course material	0	0	0	1	8	0	**4.89**	4.54	4.47	**4.41**
5. The instructor assigned grades fairly	0	0	0	1	8	0	**4.89**	4.32	4.35	**4.24**
6. The instructional methods encouraged student learning	0	0	1	0	8	0	**4.78**	4.16	4.13	**4.08**
7. I learned a great deal in this course	0	0	0	1	8	0	**4.89**	4.26	4.24	**4.10**
8. I had a strong prior interest in the subject matter and wanted to take this course	0	0	0	2	7	0	**4.78**	4.03	3.94	**3.62**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as	0	0	0	2	7	0	**4.78**	4.27	4.20	**4.11**
10. I rate the overall quality of the course as	0	0	0	0	9	0	**5.00**	4.14	4.14	**3.99**

What do you like best about this course?:

“Information was presented in an organized manner, and explanations were clear”

“the instructor made the material easier to understand”

If you were teaching this course, what would you do differently?:

“Mr. Charnley is the best TA I've had so far. He always has positive attitude towards his students and is always willing to help. One comment though: Mr. Charnley used the same examples during recitations as professor did which was not that helpful. Other then that, great teacher! Thank you”

“i would remind students the day before class to have questions ready to ask during class”

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

“he has helped me learn to take a complicated concept and break it down into simpler parts”

“TA Charnley taught us the meaning behind our math, explaining to us the proofs and applications of the theories we learned.”

Other comments or suggestions::

“Best TA I've ever had. Very helpful.”

“Thank you so much for a great semester! You made the material so much easier to understand and you made calc 3 as fun as it could possibly be.”

“you're awesome”

“Excellent job on part of the TA.”

“Awesome. Takes time to schedule office hours with students ”

Rutgers University Student Instructional Rating
(Online Survey)

Charnley Matthew Fall 2015, 01:640:251:H1 — Multivariable Calc (index #03919) Enrollment= 10, Responses= 8 **Part A: University-wide Questions:**	Student Responses Strong Disagree 1				Strong Agree 5	No response	Weighted Means Section	Course	Level	Dept
1. The instructor was prepared for class and presented the material in an organized manner.	0	0	1	0	7	0	**4.75**	4.35	4.25	**4.30**
2. The instructor responded effectively to student comments and questions.	0	0	1	0	7	0	**4.75**	4.29	4.19	**4.21**
3. The instructor generated interest in the course material.	0	0	1	0	7	0	**4.75**	4.13	4.00	**4.06**
4. The instructor had a positive attitude toward assisting all students in understanding course material.	0	0	1	0	7	0	**4.75**	4.47	4.33	**4.34**
5. The instructor assigned grades fairly.	0	0	1	1	6	0	**4.63**	4.29	4.21	**4.21**
6. The instructional methods encouraged student learning.	0	0	1	1	6	0	**4.63**	4.03	3.95	**3.98**
7. I learned a great deal in this course.	0	0	1	0	7	0	**4.75**	4.17	4.08	**3.98**
8. I had a strong prior interest in the subject matter and wanted to take this course.	0	0	1	2	5	0	**4.50**	3.95	3.87	**3.56**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as:	0	0	0	0	8	0	**5.00**	4.08	3.98	**4.00**
10. I rate the overall quality of the course as:	0	0	0	2	6	0	**4.75**	3.99	3.92	**3.90**

What do you like best about this course?:

> “While I learned virtually nothing from the lectures, I benefited greatly from the recitations.”

> “I liked that we reviewed homework problems in detail and discussed basic concepts of each section in detail. I was able to understand most topics covered.”

> “It cleared up anything I didn't understand in lecture. Was very clear and easy to follow.”

> “The recitation. It was by far the most useful tool available for understanding the material due to the instructor.”

> “This course has shown me the origins of the mathematical formulas and concepts from the Multivariable Calculus H1 lecture in a very simple manner.”

> “I liked how Matt was always positive in class and clearly explained the material.”

If you were teaching this course, what would you do differently?:

> “I would not do anything differently.”

> “nothing.”

> “I wouldn't change a thing.”

> “I would show some more applications of the mathematical concepts that allow for them.”

> “None”

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

> “The instructor helped me to fully understand the material that I did not comprehend in lecture.”

> “Encouraged good work ethic because homework is gone over in class, so it is beneficial to have it mostly completed.”

> “This course and instructor have shown me that higher level courses in mathematics can in fact tell one more about the basic roots of math itself.”

> “He made the material much easier to learn, which made me more motivated to listen and stay attentive in class.”

Other comments or suggestions::

> “Recitation was very helpful and the instructor's teaching style was clear and easy to understand. ”

> “Very effective!”

> “Overall excellent TA”

> “None”

Rutgers University Student Instructional Rating
(Online Survey)

Charnley Matthew Fall 2015, 01:640:251:H2 — Multivariable Calc (index #07513) Enrollment= 17, Responses= 8	**Student Responses**						**Weighted Means**			
Part A: University-wide Questions:	**Strong Disagree 1**				**Strong Agree 5**	**No response**	**Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner.	0	0	0	0	8	0	**5.00**	4.35	4.25	**4.30**
2. The instructor responded effectively to student comments and questions.	0	0	0	0	8	0	**5.00**	4.29	4.19	**4.21**
3. The instructor generated interest in the course material.	0	0	1	2	5	0	**4.50**	4.13	4.00	**4.06**
4. The instructor had a positive attitude toward assisting all students in understanding course material.	0	0	0	0	8	0	**5.00**	4.47	4.33	**4.34**
5. The instructor assigned grades fairly.	0	0	0	1	7	0	**4.88**	4.29	4.21	**4.21**
6. The instructional methods encouraged student learning.	0	0	2	0	6	0	**4.50**	4.03	3.95	**3.98**
7. I learned a great deal in this course.	0	0	0	2	6	0	**4.75**	4.17	4.08	**3.98**
8. I had a strong prior interest in the subject matter and wanted to take this course.	0	0	0	1	7	0	**4.88**	3.95	3.87	**3.56**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as:	0	0	0	1	7	0	**4.88**	4.08	3.98	**4.00**
10. I rate the overall quality of the course as:	0	0	1	0	7	0	**4.75**	3.99	3.92	**3.90**

What do you like best about this course?:

“This recitation was a lot more clear on the material than the lecture.”

“He explained everything so much better than the professor. Matt always made himself available to help us learn more outside of the classroom. He knows exactly what he is talking about. In one recitation I learn more than in two lectures from the professor. ”

“I liked two things: that everything was covered again so that it was better learned and that multiple approaches were shown for a few difficult things.”

“Matt was one of the best math TAs I've ever had. He was detailed and extremely clear with all the lessons that he taught. ”

“Matt was fantastic, the best TA i have had at Rutgers. ”

“Matt helped us understand the course material very well. He often made up for the instructor's shortcomings.”

“Mr. Charnley was very clear with everything he taught and was good at simplifying concepts we may not have understood at first.”

If you were teaching this course, what would you do differently?:

“N/A”

“Nothing. ”

“Maybe a few details on the more abstract things. Better coordination with the professor, if possible.”

“nothing at all”

“Nothing”

“I think it did a perfect job as a TA.”

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

“Instructor taught with clarity and even created an online review session before an exam.”

“He made calc entertaining and easy to learn. Instead of telling us the material was extremely tough he would make it very doable and provided positive encouragement. ”

“This course offered a nice tie-up of what the professor showed and made the material much more accessible.”

“Matt has really sparked my interest in mathematics and made multivariable calculus an easy to understand and fun course.”

“Gave me a different insight to Calculus not seen in the book”

“In many ways. :O”

“I wasn't really pushed towards growth outside the class, but I don't really see a problem with that in terms of this class.”

Other comments or suggestions::

“He should be a professor. ”

“Matt was a fantastic TA. :)”

“Mr. Charnley did a fantastic job. He was very helpful and knowledgeable, and very personable.”

Rutgers University Student Instructional Rating
(Online Survey)

Charnley Matthew Fall 2015, 01:640:251:H3 — Multivariable Calc (index #09825) Enrollment= 9, Responses= 6 **Part A: University-wide Questions:**	**Student Responses** **Strong Disagree 1**				**Strong Agree 5**	**No response**	**Weighted Means** **Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner.	0	0	0	1	5	0	**4.83**	4.35	4.25	**4.30**
2. The instructor responded effectively to student comments and questions.	0	0	0	0	6	0	**5.00**	4.29	4.19	**4.21**
3. The instructor generated interest in the course material.	0	0	1	1	4	0	**4.50**	4.13	4.00	**4.06**
4. The instructor had a positive attitude toward assisting all students in understanding course material.	0	0	0	1	5	0	**4.83**	4.47	4.33	**4.34**
5. The instructor assigned grades fairly.	0	0	0	0	6	0	**5.00**	4.29	4.21	**4.21**
6. The instructional methods encouraged student learning.	0	0	0	2	4	0	**4.67**	4.03	3.95	**3.98**
7. I learned a great deal in this course.	0	0	1	1	4	0	**4.50**	4.17	4.08	**3.98**
8. I had a strong prior interest in the subject matter and wanted to take this course.	0	0	2	2	2	0	**4.00**	3.95	3.87	**3.56**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as:	0	0	0	0	6	0	**5.00**	4.08	3.98	**4.00**
10. I rate the overall quality of the course as:	0	0	0	1	5	0	**4.83**	3.99	3.92	**3.90**

What do you like best about this course?:

"He clearly explained lecture material as well as how to solve difficult homework problems. It was very helpful. "

"The teacher used colored chalk to draw easily understandable pictures, took all of our homework questions, and covered all of the material that was covered in lecture clearly and in a different way than the professor, solidifying and enhancing our understanding."

"I liked resuscitation the best because it's awesome!"

"I liked that it corresponded well with what I was learning in Physics class."

"I liked how everything was always explained very clearly with graphs usually and there were never any tricks on the quizzes."

If you were teaching this course, what would you do differently?:

"n/a"

"Nothing."

"More Michigan jokes."

"More communication between the professor and the TA"

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

"I can understand how certain formulas and theorems were derived, so I understand the material much better. "

"The instructor did a great job in answering questions from the homework and explained the subject matter in a very clear and understandable way that allowed me to better understand the subject."

"Class was not only informative and interesting but fun as well, making me excited for calculus class each week."

"The instructor has encouraged us to go on and use multivariable calculus to the best of our abilities! Thanks!"

"Mr. Charnley is really knowledgeable and does his best to help us fully understand the topics presented in class. He is so prepared that he even has the difficult homework problems memorized. He has also brought in colored chalk that really helped us picture the curves and surfaces we were supposed to integrate."

Other comments or suggestions::

"Use more colored chalk, it helps a lot."

":)"

"Thank you very much for teaching me this semester"

Rutgers University Student Instructional Rating
(Online Survey - Sakai)

Charnley Matthew mpc163 Fall 2017, 01:640:421:04 — ADV CALC FOR ENGRNG (index #05351) Enrollment= 87, Responses= 32	**Student Responses**						**Weighted Means**			
Part A: University-wide Questions:	**Strong Disagree 1**				**Strong Agree 5**	**No response**	**Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner.	0	0	1	6	17	8	**4.67**	4.30	4.27	**4.34**
2. The instructor responded effectively to student comments and questions.	0	0	1	6	20	5	**4.70**	4.27	4.26	**4.22**
3. The instructor generated interest in the course material.	0	0	4	4	19	5	**4.56**	4.08	4.10	**4.05**
4. The instructor had a positive attitude toward assisting all students in understanding course material.	0	0	1	7	21	3	**4.69**	4.44	4.40	**4.33**
5. The instructor assigned grades fairly.	0	1	6	3	14	8	**4.25**	4.14	4.19	**4.17**

6. The instructional methods encouraged student learning.	0	0	2	7	18	5	**4.59**	4.16	4.07	**3.99**
7. I learned a great deal in this course.	1	0	6	5	17	3	**4.28**	4.12	4.11	**4.02**
8. I had a strong prior interest in the subject matter and wanted to take this course.	0	2	6	6	17	1	**4.23**	3.91	4.05	**3.61**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as:	0	0	4	6	17	5	**4.48**	4.08	4.07	**4.02**
10. I rate the overall quality of the course as:	0	1	6	10	12	3	**4.14**	3.99	3.99	**3.90**
Part B: Questions Added by Department or Instructor										
	Strongly disagree	**Disagree**	**Uncertain**	**Agree**	**Strongly agree**					
15. The TA was a valuable resource for this course.	0	0	2	3	22	5	**4.74**	-	-	**-**
16. The online office hours were a valuable resource for this course.	0	0	5	0	18	9	**4.57**	-	-	**-**
17. The recordings of the online office hours were helpful for this course.	0	0	4	3	16	9	**4.52**	-	-	**-**

What do you like best about this course?:

“this is interesting.”

“I enjoyed the earlier material and the application of using Laplace Transform in solving differential equations. ”

“Matt is the best TA I have ever had and is one of the smartest people I have ever talked to. No matter how poorly you word a question, Matt somehow knows what you mean and has an answer that goes above and beyond. Matt effortlessly guides students from no understand and complete confusion to clarity and competence. Matt has been by far the best thing about the class and I wish him the all the best in the future.”

“I did not attend any office hours or reach out to Matthew Charnley (TA). However, from all the emails he has sent and trying to keep everyone updated on how they can contact him for more help shows that he was dedicated to trying to aid the students. ”

“I liked how the material was presented in a straightforward way.”

“I liked all of the review sessions.”

“N/A - See Terrence Butler”

“Topics toward the end of the class were fairly interesting”

“It deals with various information that can be applied in real life.”

“N/A”

“The TA is very knowledgeable about the subject and tries his best to help the students the best he can.”

“The time of the course ”

If you were teaching this course, what would you do differently?:

“I think this proffesor’s teaching is pretty effective i don’t want to do something different. ”

“I would make the latter half of the material presented in a manner which was more easily understood. There seemed to be a lot of different cases for solving Boundary Value Problems which kind somewhat blended together in terms of applying different techniques. ”

“offer recitation to some sections at least ”

“I think he did a good job from what I could tell. As stated above, I did not get to experience his teaching style first hand. ”

“I would not change anything about the course.”

“Recitations would have been helpful.”

“It would have been nice to see Matt in class more because he was always so helpful (also so that he can actually keep up with what we are doing in the class so Butler doesn't surprise him)”

“How homework was graded”

“Have everything in the class build upon each other.”

“N/A”

“There was a lack of communication between the instructor and the TA which lead to uncertainties regarding the material on the exams.”

“I would give extra credit for group study ”

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

“he give full attention in iffice hours and help to deal with challenges we face in this course.”

“No matter how hard Rutgers makes me want to hate math, Matt keeps that light alive because he is terrific.”

“Matthew Charnley has not encouraged my intellectual growth or progress. This is not due to his style of teaching but more because of the lack of meetings I have had with him. ”

“He is always so enthusiastic about everything and he always has a smile on his face. I enjoy being in the same room as him since he truly does bring a positive energy wherever he goes. ”

“Always helpful and showed me the "behind the scenes" of how we could get to certain conclusions, made me more interested in math”

“None”

“The enthusiasm of the instructor made the class more interesting.”

“N/A”

“Tough grader”

Other comments or suggestions::

“overall excellent professor.”

“Matthew Charnley seemed enthusiastic and flexible to the students. He was very active and I am sure that carried over to his teaching style. ”

“Great guy.”

“Keep doing the workshop and review sessions, they are very helpful”

“One of the best TA's that I've had so far at Rutgers”

“Professor shouldn’t make mistakes in class to much. Students are supposed to ”

What was your level of interaction with the TA?:

“Average”

“not that bad”

“I did not go to any of the TA's office hours, nor did I participate in the online office hours, although I did look at some of the notes which he posted and attempted to watch some of the online office hours after they had been conducted for further insight into topics which I saw were covered in that office hour.”

“Fairly often”

“frequent ”

“I never went to office hours for this course.”

“I rarely interacted with the TA. My class/work schedule conflicted with the TA's availability.”

“Office hours fairly often”

“Occasional office hours and review sessions”

“None”

“high”

“Moderate”

“Occasionally asked him a question related to a concept or homework question outside of class”

“Very minimal but good when I had to.”

“Almost none, except for review sessions before exams”

“Went almost every week”

“10/10”

Do you think there should be a TA for this course? Why or why not?:

“Yes, extra help is always great. Also, students can get a different explanation or examples in order to better understand the problem.”

“i don’t think TA is needed for this course.”

“I think there definitely needs to be a TA for the course because it does get very complicated and I think there needs to be someone to go to besides the professor to get assistance form. Professor Butler is also quite old and therefore may not be as reliable during his office hours as some of the TAs would be”

“YES ”

“Yes I think the TA was useful to many students.”

“I personally did not require a TA for this course. However, a TA in any math course should be required. Students have different learning preferences. Some may not find lectures effective but much rather 1 on 1 or small group learning environments which is why a TA proves useful in this course. ”

“Yes because the professor was awful”

“Yes, very helpful for homework questions and to get a second way to explain how to approach the problems”

“Yes because there should definitely be someone else to teach the material besides the professor since the professor makes so many mistakes in every example he does. ”

“yes, to help you understand the material you don't get from the lecture. ”

“Yes. Review sessions were helpful! It was nice having additional resources.”

“Absolutely, Matt was an invaluable resource and was always helpful at explaining things that could not be cleared up from the class or book”

“Yes because the professor may be busy dealing with other students and work.”

“Yes, there should be a TA. The exam review sessions are very helpful.”

“Yes, because the TA's sometimes explain concepts in a way that the professor can't, and the way the TA explains it just clicks.”

“Yes, it is very helpful and is a necessity. ”

“They should because the professor can’t teach ”

Rutgers University Student Instructional Rating
(Online Survey - Sakai)

Charnley Matthew mpc163 Fall 2017, 01:640:421:05 — ADV CALC FOR ENGRNG (index #09872) Enrollment= 82, Responses= 22	**Student Responses**						**Weighted Means**			
Part A: University-wide Questions:	**Strong Disagree 1**				**Strong Agree 5**	**No response**	**Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner.	0	0	3	4	9	6	**4.38**	4.30	4.27	**4.34**
2. The instructor responded effectively to student comments and questions.	0	0	3	3	12	4	**4.50**	4.27	4.26	**4.22**
3. The instructor generated interest in the course material.	0	0	4	8	6	4	**4.11**	4.08	4.10	**4.05**
4. The instructor had a positive attitude toward assisting all students in understanding course material.	0	0	3	3	13	3	**4.53**	4.44	4.40	**4.33**
5. The instructor assigned grades fairly.	0	0	3	7	8	4	**4.28**	4.14	4.19	**4.17**

6. The instructional methods encouraged student learning.	0	0	4	6	8	4	**4.22**	4.16	4.07	**3.99**
7. I learned a great deal in this course.	0	1	5	5	8	3	**4.05**	4.12	4.11	**4.02**
8. I had a strong prior interest in the subject matter and wanted to take this course.	0	2	7	3	6	4	**3.72**	3.91	4.05	**3.61**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as:	0	0	3	7	8	4	**4.28**	4.08	4.07	**4.02**
10. I rate the overall quality of the course as:	0	1	6	6	6	3	**3.89**	3.99	3.99	**3.90**
Part B: Questions Added by Department or Instructor										
	Strongly disagree	**Disagree**	**Uncertain**	**Agree**	**Strongly agree**					
15. The TA was a valuable resource for this course.	0	1	3	3	11	4	**4.33**	-	-	**-**
16. The online office hours were a valuable resource for this course.	1	1	7	3	4	6	**3.50**	-	-	**-**
17. The recordings of the online office hours were helpful for this course.	1	0	8	3	4	6	**3.56**	-	-	**-**

What do you like best about this course?:

"Good sense of humor with a very positive attitude toward students."

"As a lover of calculus, I found the topics very interesting "

"Matt gives the most practical essence of the course"

"The organization of material"

"He held review sessions before our exams, which allowed us to go over key problems that we had questions on. This greatly helped prepare us for the exam."

If you were teaching this course, what would you do differently?:

"Nothing."

"Be more accessible to the students "

"Do it more often, I wish this course had a recitation"

"More help regarding PDEs "

"I would not do anything differently."

In what ways, if any, has this course or the instructor encouraged your intellectual growth and progress?:

"None."

"It has not "

"None"

"The TA was very knowledgeable about the material and helped answer our questions. "

Other comments or suggestions::

"I didn't see the ta at all this semester "

"NA"

What was your level of interaction with the TA?:

"None."

"Didn't know him "

"I didn't really need to go see the TA for anything but when we came into class a few times, he seemed to want to make himself as available as possible. He offered online resources and a lot of extra help which I thought was a very good idea for this class."

"went to the ta before the test"

"Went to the review sessions"

"Attended review sessions for exams I and II and the final exam workshops"

"I attended the exam review sessions and they were pretty helpful. "

"Frequent during exams"

"I didn't go to office hours, but I went to all the review sessions and found them very helpful. "

“I went to his review sessions for the exams.”

“very low; I have never seen him in action because I do not go to his office hours.”

“not a lot”

Do you think this course should have a TA? Why or why not?:

“Yes, it helps students digest what they learned.”

“Yes. Easier access to one on one help.”

“It should have a recitation where the ta teaches the material ”

“Yes. The material is difficult sometimes and if a student needed help, having a TA would definitely help”

“I think the TA should be more involved”

“Yes, this course should have a TA. ”

“Yes, this course should have recitations.”

“Yes because sometimes the professor can't cover everything in a clear way and the TA is great to answer any questions that the students might have. ”

“Yes. The TA taught me more material than the professor. ”

“Yes, there should be a TA for the class. Having an extra person that can explain concepts differently from the professor is very helpful and useful. The TA provides a different perspective and approach to problems that are helpful for topics we do not fully understand.”

“Yes, considering that a lot of people in this course asks for help all the time.”

“Doesn't really matter for me tbh. I never went to any office hour etc. I'm doing fine. ”

Rutgers University Student Instructional Rating
(Online Survey - Sakai)

Charnley Matthew Spring 2018, 16:642:574:01 — Numerical Analysis II (index #05056) Enrollment= 21, Responses= 10 **Part A: University-wide Questions:**	**Student Responses**						**Weighted Means**			
	Strong Disagree 1				**Strong Agree 5**	**No response**	**Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner.	0	0	1	0	9	0	**4.80**	4.90	4.76	**4.77**
2. The instructor responded effectively to student comments and questions.	0	0	0	0	10	0	**5.00**	5.00	4.82	**4.83**
3. The instructor generated interest in the course material.	0	0	1	0	9	0	**4.80**	4.89	4.70	**4.71**
4. The instructor had a positive attitude toward assisting all students in understanding course material.	0	0	0	1	9	0	**4.90**	4.95	4.76	**4.77**
5. The instructor assigned grades fairly.	0	0	0	0	10	0	**5.00**	5.00	4.78	**4.78**
6. The instructional methods encouraged student learning.	0	0	0	0	10	0	**5.00**	5.00	4.73	**4.73**
7. I learned a great deal in this course.	0	0	1	0	9	0	**4.80**	4.90	4.61	**4.62**
8. I had a strong prior interest in the subject matter and wanted to take this course.	0	0	0	1	9	0	**4.90**	4.90	4.66	**4.67**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as:	0	0	0	1	9	0	**4.90**	4.90	4.73	**4.73**
10. I rate the overall quality of the course as:	0	0	1	0	9	0	**4.80**	4.85	4.69	**4.69**

What do you like best about this course?:

"Matt is very nice and his solution is very explicit and easy to understand. He really put a lot of effort to the course"

If you were teaching this course, what would you do differently?:

"I may be trying to communicate with the professor more and trying to know what professor should expect us to learn and what the exam will look like etc."

Other comments or suggestions::

"Matt helps me a lot in the problem sessions.Hope you can tell more about Matlab coding."

Rutgers University Student Instructional Rating
(Online Survey - Sakai)

Charnley Matthew Spring 2018, 16:642:575:01 — Numerical Solutions of Partial Differential Equati (index #06743) Enrollment= 19, Responses= 4 **Part A: University-wide Questions:**	**Student Responses** **Strong Disagree 1**				**Strong Agree 5**	**No response**	**Weighted Means** **Section**	**Course**	**Level**	**Dept**
1. The instructor was prepared for class and presented the material in an organized manner.	0	0	1	2	1	0	**4.00**	4.14	4.76	**4.77**
2. The instructor responded effectively to student comments and questions.	0	0	0	2	2	0	**4.50**	4.29	4.82	**4.83**
3. The instructor generated interest in the course material.	0	0	0	2	2	0	**4.50**	4.29	4.70	**4.71**
4. The instructor had a positive attitude toward assisting all students in understanding course material.	0	0	0	2	2	0	**4.50**	4.14	4.76	**4.77**
5. The instructor assigned grades fairly.	0	0	0	3	1	0	**4.25**	4.14	4.78	**4.78**
6. The instructional methods encouraged student learning.	0	0	0	3	1	0	**4.25**	3.86	4.73	**4.73**
7. I learned a great deal in this course.	0	0	1	2	1	0	**4.00**	3.86	4.61	**4.62**
8. I had a strong prior interest in the subject matter and wanted to take this course.	0	0	1	2	1	0	**4.00**	4.00	4.66	**4.67**
	Poor				**Excellent**					
9. I rate the teaching effectiveness of the instructor as:	0	0	0	2	2	0	**4.50**	4.29	4.73	**4.73**
10. I rate the overall quality of the course as:	0	0	0	3	1	0	**4.25**	4.00	4.69	**4.69**

Other comments or suggestions::

> "For the first half of the semester there seemed to be a bit of a disconnect between teacher and teacher's assistant. Once Matt took over the class it improved, as he knew what was going on in both."
