Clairaut's Theorem

Matt Charnley

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In this document, we describe a function that does not satisfy the hypotheses of Clairaut's theorem, and therefore, the mixed second partial derivatives are not equal.

Let

$$f(x,y) = \frac{xy(x^2 - y^2)}{x^2 + y^2}$$

with $f(0,0) = 0$. We compute the first partial derivatives

$$f_x(x,y) = \begin{cases} \frac{y(x^2-y^2)+2x^2y}{(x^2+y^2)} - \frac{2x^2y(x^2-y^2)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

and

$$f_y(x,y) = \begin{cases} \frac{x(x^2-y^2)+2xy^2}{(x^2+y^2)} - \frac{2xy^2(x^2-y^2)}{(x^2+y^2)^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

where we needed to look at the actual function f and take limits to get the value of these derivatives at $(0,0)$. To compute the second derivatives at $(0,0)$ we follow this same approach.

$$f_{xy}(0,0) = \lim_{h \to 0} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h \to 0} \frac{(-h+0) - 0}{h} = -1$$

since, plugging $x = 0$ into the equation for f_x gives

$$f_x(0,y) = \frac{y(0 - y^2) + 0}{0 + y^2} - 0 = -y.$$

Doing the same thing for f_{yx} we see that

$$f_{yx}(0,0) = \lim_{h \to 0} \frac{f_y(h,0) - f_y(0,0)}{h} = \lim_{h \to 0} \frac{(h+0) - 0}{h} = 1.$$

Thus, we have that $f_{xy} \neq f_{yx}$ at the point $(0,0)$. This is because neither of these derivatives are continuous in a disk containing $(0,0)$.