Math 152 - Worksheet 11

Section 7.7 - Improper Integrals

Learning Problems

These problems should be completed on your own. If you need hints on solving a problem, there are some provided with each problem. These are provided on the following pages, with one 'level' of hint per page, with the earlier ones giving away less of the problem than the later ones. Try to work from the earlier hints to the later ones, as this will give you the practice you need to succeed in this class.

1. Determine if $\displaystyle\int_0^6 \frac{1}{x^{10/11}}\,dx$ converges, and if so, evaluate it.
2. Determine if $\int_{-3}^4 \frac{1}{x^2}\,dx$ converges, and if so, evaluate it.
3. Determine if $\int_{-\infty}^{\infty} \frac{x}{1+x^2}\,dx$ converges, and if so, evaluate it.
4. Determine if $\displaystyle\int_5^{\infty} \frac{2}{(x-1)(x+3)}\,dx$ converges, and if so, evaluate it.
5. Determine if $\int_2^{\infty} \frac{1}{x^4+x}\,dx$ converges and explain why.
6. Determine if $\displaystyle\int_4^{\infty} \frac{1}{x^6-x}\,dx$ converges and explain why.

Submission Problems

1. Determine if $\int_{-3}^2 \frac{1}{x^{2/3}}\,dx$ converges, and if so, evaluate it.
2. Determine if $\int_{-\infty}^{\infty} e^{-x^2}\,dx$ converges. Hint: Compare with e^x if $x < -1$ and e^{-x} if $x > 1$. Provide the explanation as to why this is the case.

Hint #1

1. Which part of the integral is improper?
2. Why is the integral improper?
3. We have infinity on both endpoints here. What does that mean for evaluating this integral?
4. This integral is improper because of the ∞ at the upper endpoint.
5. We don't have a way to evaluate this integral directly, so what else can we do to determine if this integral converges?
6. This is another integral that we can not evaluate directly, so we need the comparison test for this.

Hint #2

1. We replace the 0 by a parameter R, and then do the integral, and limit $R \to 0$
2. There is a vertical asymptote at 0, so we need to split the integral there to solve it.
3. We need to split this integral into two pieces, and zero is the best place to do so.
4. How do we evaluate $\int_5^R \frac{2}{(x-1)(x+3)}\,dx$? Partial Fractions.
5. We need to use the comparison test. Based on the form of the expression should this converge or diverge?
6. Referring to the last problem, we could try to use $\frac{1}{x^6}$ as a comparison. However, $x^6 - x < x^6$, not the other way around. What else can we do?

Hint #3

1. The limit we then need to compute is $\lim_{R\to 0} 11(6^{1/11}) - 11(R^{1/11})$
2. Once we split the integral, we need to take the limits $\displaystyle\lim_{R\to 0^-} \frac{1}{R} + \frac{1}{3}$ and $\displaystyle\lim_{R\to 0^+} \frac{1}{4} - \frac{1}{R}$
3. To evaluate $\displaystyle\int_0^\infty \frac{x}{1+x^2}\,dx$, use the u-substitution $u = 1 + x^2$.
4. The decomposition is $\frac{1/2}{x-1} - \frac{1/2}{x+3}$, which integrates to $\frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+3|$.
5. The x^4 tells me that this should converge. Since $x^4 + x \leq x^4$ for all $x \geq 2$, what can we compare to?
6. Since we are looking at $x > 4$, we know that $x^5 > 4$ (yes, we could do a lot better, but this is enough), so that $x^6 = x^5 x > 4x$, so that $x < \frac{1}{4}x^6$. This implies that $x^6 - x \geq x^6 - \frac{1}{4}x^6 = \frac{3}{4}x^6$. How does that help?

Hint #4

3. Each of the two integrals becomes $\int_1^\infty \frac{1}{2u}\, du$
4. The limit we need to compute is $\lim_{R\to\infty} \frac{1}{2}\ln\left|\frac{R-1}{R+3}\right| - \frac{1}{2}\ln\left|\frac{5-1}{5+3}\right|$
5. Use the function $\frac{1}{x^4}$, whose integral converges, as a comparison.
6. Do comparison test with the function $\frac{4}{3}\frac{1}{x^6}$

Answers

1. The integral converges to $116^{1/11}$.
2. The integral diverges.
3. The integral diverges.
4. This integral converges to $-\frac{1}{2}\ln\frac{1}{2}$.
5. Since $\frac{1}{x^4+x} \leq \frac{1}{x^4}$ and $\displaystyle\int_2^\infty \frac{1}{x^4}\,dx$ converges, we know that $\displaystyle\int_2^\infty \frac{1}{x^4+x}\,dx$ converges.
6. Since $x > 4$, we have that $x^6 > 4x$. Therefore, we know that $\frac{1}{x^6-x} \leq \frac{4}{3}\frac{1}{x^6}$ and $\displaystyle\int_4^\infty \frac{1}{x^6}\,dx$ converges, we know that $\displaystyle\int_4^\infty \frac{1}{x^6-x}\,dx$ converges.