Taylor Polynomials

Learning Goals

- Find the Taylor polynomials for a function at a value
- Find the Maclaurin polynomials for a function
- Determine the error of a estimated function value using Taylor's theorem

Contents

1 Truncating Taylor Series

There are two ways to approach the idea of Taylor Polynomials. The first of these is by truncating Taylor Series, or cutting them off after some number of terms.

Say you want to use the Taylor Series to approximate a function on a computer. If you want to program it in, you can only use a finite number of terms, not the entire series. So, if you only use a finite number of terms, is that still a good approximation? The answer is yes.

Definition. For a function f that has at least n derivatives at a, we define the *Taylor Polynomial of order n of f at a* is

$$T_n(x) = \sum_{k=0}^{n} \underbrace{\frac{f^{(k)}(a)}{k!}(x-a)^k}_{\text{Taylor Series terms}}$$

$$T_2(x) = \frac{f^{(0)}(a)}{0!}(x-a)^0 + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

Maclaurin Polynomial : Center a is 0.

We can use the things we have learned from Taylor Series in the last section to write out some Taylor Polynomials.

Example: Write out the 4th order Taylor Polynomial of $\frac{1}{1-x}$ centered at 0.

Taylor Series

$$\frac{1}{1-x} = \underbrace{\sum_{n=0}^{\infty} x^n}$$

$$T_4(x) = 1 + x + x^2 + x^3 + x^4$$

2 Extension of Linear Approximation

The other way to think about Taylor Polynomials is as an extension of Linear Approximation. If we have a function f, what is the first order Taylor Polynomial of f centered at a?

$$T_1(x) = f(a) + f'(a)(x-a)$$

→ Tangent Line

→ Linear Approximation

Well, what if I want to do better than linear approximation? What is the best-fit quadratic to my function? What about the best fit 4th-order polynomial? We get all of these from Taylor Polynomials.

Theorem. *The nth-order Taylor Polynomial of f and a, $T_n(x)$ is the only nth degree polynomial that matches f and all of its first n derivatives at a.*

$$a_k = \underbrace{\frac{f^{(k)}(0)}{k!}}$$

$$\frac{f'''(0)}{3!}x^3 \rightsquigarrow f'''(0)$$

→ No other polynomial matches better than $T_n(x)$.

Example: Find the best-fit cubic (degree 3) polynomial to the function $f(x) = e^{2x}$ around the point $x = 1$.

$$T_3(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3$$

$f(x) = e^{2x}$ $\quad f(1) = e^2$

$f'(x) = 2e^{2x}$ $\quad f'(1) = 2e^2$

$f''(x) = 4e^{2x}$ $\quad f''(1) = 4e^2$

$f'''(x) = 8e^{2x}$ $\quad f'''(1) = 8e^2$

$$T_3(x) = e^2 + 2e^2(x-1) + \frac{\overset{2}{\cancel{4}}e^2}{\cancel{2}}(x-1)^2 + \frac{\overset{4}{\cancel{8}}e^2}{\underset{3}{\cancel{6}}}(x-1)^3$$

3 Error Bound

$T_n(x)$ is the best-fit nth degree polynomial to the function f at a. How close is it? This comes from the error bound on Taylor Polynomials.

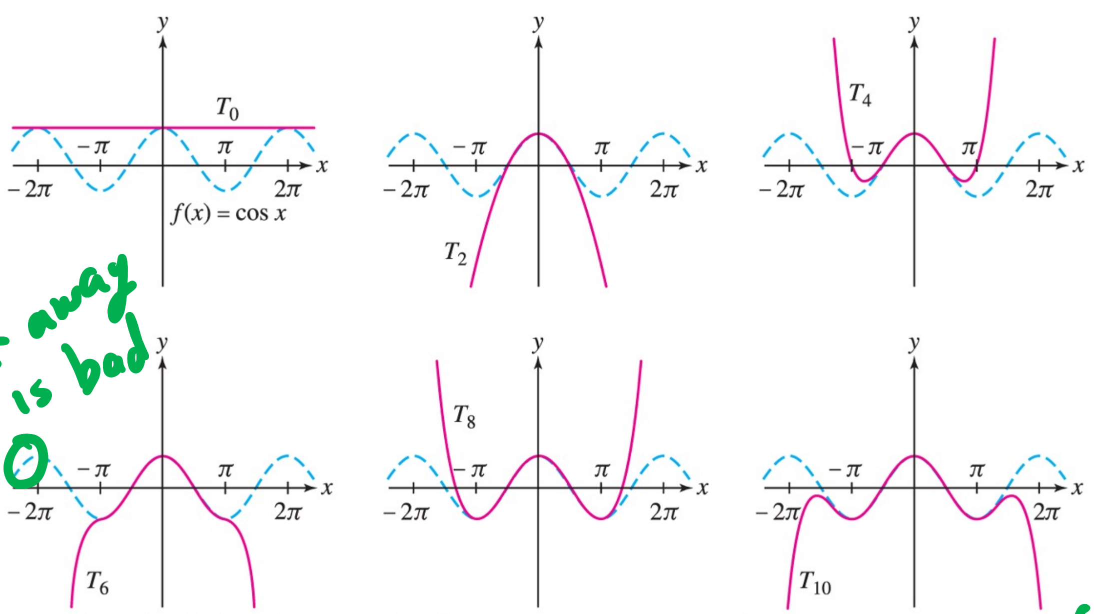
Theorem. *Assume that $f^{(n+1)}$ exists and is continuous. Let K be a number so that $|f^{(n+1)}(u)| \leq K$ for all u between x and a. Then*

$$\boxed{|f(x) - T_n(x)| \leq K\frac{|x-a|^{n+1}}{(n+1)!}}$$

where T_n is the nth Taylor Polynomial of f at a.

- If x is really far from a, this is a bad approximation.
- More terms gives a better approximation.

Far away is bad



Rogawski et al., *Calculus: Early Transcendentals*, 4e, © 2019 W. H. Freeman and Company

- More terms work for a wider range.

Example: Find the error in approximating ln(1.5) by its 4th order Taylor Polynomial centered at 1.

$$|f(x) - T_4(x)| \le K \frac{|x-a|^5}{5!}$$

$x = 1.5 \quad a = 1$

$$|f(1.5) - T_4(1.5)| \le K \frac{(.5)^5}{5!}$$

$|f^{(5)}(u)| \le K$ for u between x and a

$f(x) = \ln(x)$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2}$

$f^{(3)}(x) = \frac{2}{x^3}$

$f^{(4)}(x) = -\frac{6}{x^4}$

$f^{(5)}(x) = \frac{24}{x^5}$

Between 1 and 1.5

Take $K = 24$

$$|\ln(1.5) - T_4(1.5)| \le \frac{24 (.5)^5}{5!} = \frac{1}{160}$$

$.00625$

4 Using this Error Bound

We will usually see the error bound written in the following form: Let $R_n(x)$ be the remainder when approximating $f(x)$ by $T_n(x)$. Then

$$|R_n(x)| \leq \underline{K_{n+1}} \frac{|x-a|^{n+1}}{(n+1)!}$$

where K_{n+1} is an upper bound for the $n+1$ derivative of f between x and a.

$R_n(x) = f(x) - T_n(x)$

K_{n+1} because it could change with different values of n.

→ Could cause this to not be a good approximation with more terms.

Proof: Bound R_n using Integration by Parts and K_{n+1}.

We can use this to finish up our statement about convergence of Taylor Series.

Theorem. *Let $I = (c-R, c+R)$, where $R > 0$ and assume that f is infinitely differentiable on I. Suppose there exists $K > 0$ such that all derivatives of f are bounded by K on I, that is*

$$|f^{(k)}(x)| \leq K$$

for all $k \geq 0$ and all x in I. Then f is represented by its Taylor Series in I:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$$

- ALL Derivatives bounded by K.

→ K_{n+1} does not change with n!!

"Proof" In order to show that the Taylor Series converges, we need to show that the remainder goes to zero as $n \to \infty$. We know the remainder can be bounded by

$$|R_n(x)| \leq K_{n+1} \frac{|x-a|^{n+1}}{(n+1)!}$$

If $T_n(x) \to f(x)$ as $n \to \infty$, then

$R_n(x) \to 0$ as $n \to \infty$

$$|R_n(x)| \leq K_{\cancel{n+1}} \frac{|x-a|^{n+1}}{(n+1)!}$$

$$\lim_{n\to\infty} |R_n(x)| \leq \lim_{n\to\infty} K \frac{|x-a|^{n+1}}{(n+1)!} \to 0$$

$$\frac{K \frac{|x-a|^{n+2}}{(n+2)!}}{K \frac{|x-a|^{n+1}}{(n+1)!}} = \frac{|x-a|}{n+2} \to 0$$

$$\lim_{n\to\infty} \frac{R^n}{n!} = 0$$

goes through squeeze thm.