

MATH 251: Quiz 5

July 9, 2015

Name: Solutions

1. Determine if the following vector fields are conservative. If so, find a potential function.

(a) $\vec{F} = \langle ye^{xy} + y^3, xe^{xy} + 3xy^2, z^4 \rangle$.

(b) $\vec{G} = \langle x^2 + 3xy, y^4 - 3xyz, e^{xz} + 3xy \rangle$.

$$(a) \quad \frac{\partial F_1}{\partial y} = xye^{xy} + 3y^2 + e^{xy} \frac{\partial F_1}{\partial z} = 0 \quad \frac{\partial F_2}{\partial z} = 0$$

$$\frac{\partial F_2}{\partial x} = xye^{xy} + 3y^2 + e^{xy} \frac{\partial F_2}{\partial x} = 0 \quad \frac{\partial F_3}{\partial y} = 0$$

$$\int F_1 dx = e^{xy} + xy^3$$

$$\int F_2 dy = e^{xy} + xy^3$$

$$\int F_3 dz = \frac{1}{5} z^5$$

$$V = e^{xy} + xy^3 + \frac{1}{5} z^5$$

(b) $\frac{\partial F_1}{\partial y} = 3y$

$\frac{\partial F_2}{\partial x} = -3yz$

\Rightarrow Not conservative.

2. Let $\vec{c}(t) = \langle t^2, t^3 \rangle$ for $0 \leq t \leq 3$.

(a) Compute $\int_C x \, ds$.

(b) Compute $\int_C \langle y^2, x+y \rangle \cdot d\vec{s}$.

$$\vec{c}'(t) = \langle 2t, 3t^2 \rangle$$

$$\begin{aligned} \|\vec{c}'(t)\| &= \sqrt{4t^2 + 9t^4} \\ &= t\sqrt{9t^2 + 4} \end{aligned}$$

$$(a) \int_C x \, ds = \int_0^3 t^2 \cdot t\sqrt{9t^2 + 4} \, dt$$

$$u = 9t^2 + 4$$

$$t^2 = \frac{u-4}{9}$$

$$du = 18t \, dt$$

$$= \frac{1}{18} \int \frac{u-4}{9} \cdot u^{1/2} \, du$$

$$= \frac{1}{162} \int_4^{85} u^{3/2} - 4u^{1/2} \, du$$

$$= \frac{1}{162} \left[\frac{2}{5} u^{5/2} - \frac{8}{3} u^{3/2} \right]_4^{85}$$

$$= \frac{1}{162} \left(\frac{2}{5} 85^{5/2} - \frac{8}{3} 85^{3/2} - \frac{2}{5} \cdot 32 + \frac{8}{3} \cdot 8 \right)$$

$$b) \int_0^3 \langle t^6, t^4 t^3 \rangle \cdot \langle 2t, 3t^2 \rangle \, dt = \int_0^3 2t^7 + 3t^4 + 3t^5 \, dt$$

$$= \frac{1}{4} t^8 + \frac{3}{5} t^5 + \frac{1}{2} t^6 \Big|_0^3 = \frac{3^8}{4} + \frac{3^6}{5} + \frac{3^6}{2} = \frac{3^8}{4} + \frac{7 \cdot 3^6}{10}$$

3. Let the surface S be the half of the sphere $x^2 + y^2 + z^2 = 1$ with $z \geq 0$. This surface can be parametrized by

$$G(u, v) = \langle \cos(v) \sin(u), \sin(v) \sin(u), \cos(u) \rangle$$

for $0 \leq v \leq 2\pi$ and $0 \leq u \leq \frac{\pi}{2}$.

- (a) Find the upward normal vector \vec{n} to this surface (as a function of u and v).
 (b) Integrate $f(x, y, z) = z$ over the surface.
 (c) Find the flux of the vector field $\vec{F}(x, y, z) = \langle -y, x, z \rangle$ through this sphere, with the upward normal vector.

$$T_u = \langle \cos v \cos u, \sin v \cos u, -\sin u \rangle$$

$$T_v = \langle -\sin v \sin u, \cos v \sin u, 0 \rangle$$

$$\vec{n} = \langle \cancel{\sin u} + \cos v \sin^2 u, \sin v \sin^2 u, \cos u \sin u \rangle$$

$$= \sin u \langle \cos v \sin u, \sin v \sin u, \cos u \rangle \checkmark$$

$$\|\vec{n}\| = |\sin u|$$

$$(b) \int_0^{\pi/2} \int_0^{2\pi} \cos u \sin u \, dv \, du = 2\pi \cdot \frac{\sin^2 u}{2} \Big|_0^{\pi/2} = \pi$$

$$(c) \int_0^{\pi/2} \int_0^{2\pi} \langle -\sin(v) \sin(u), \cos v \sin u, \cos u \rangle \cdot \sin u \langle \cos v \sin u, \sin v \sin u, \cos u \rangle \, dv \, du$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \sin u \cdot (-\cancel{\sin v \cos v} \sin^2 u + \cancel{\cos v \sin v} \sin^2 u + \cos^2 u) \, dv \, du$$

$$= \int_0^{\pi/2} \int_0^{2\pi} \cos^2 u \sin u \, dv \, du = 2\pi \cdot \frac{-\cos^3 u}{3} \Big|_0^{\pi/2} = \frac{2\pi}{3}$$