MATH 251: Quiz 4

June 25, 2015

Name: Solutions

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1. Integrate $f(x,y,z) = 2x + 3yz$ over the rectangular prism $0 \le x \le 2$, $1 \le y \le 5$, $0 \le z \le 1$.

$$\int_0^2\int_1^5\int_0^1 2x+3yz\,dz\,dy\,dx$$

$$=\int_0^2\int_1^5 2xz+\frac{3}{2}yz^2\Big|_0^1\,dy\,dx$$

$$=\int_0^2\int_1^5 2x+\frac{3}{2}y\,dy\,dx=\int_0^2 2xy+\frac{3}{4}y^2\Big|_1^5\,dx$$

$$=\int_0^2 8x+\underbrace{\frac{3}{4}(25-1)}_{18}\,dx=4x^2+18x\Big|_0^2$$

$$=16+36=\boxed{52}$$

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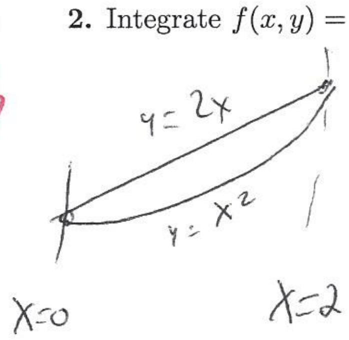
2. Integrate $f(x,y) = 2xy$ over the region between the graphs of $y = 2x$ and $y = x^2$.



$$\int_0^2\int_{x^2}^{2x} 2xy\,dy\,dx$$

$$=\int_0^2 xy^2\Big|_{x^2}^{2x}\,dx$$

$$=\int_0^2 x(2x)^2-x(x^2)^2\,dx$$

$$=\int_0^2 4x^3-x^5\,dx=x^4-\frac{x^6}{6}\Big|_0^2$$

$$=16-\frac{64}{6}=16-\frac{32}{3}$$

$$=\boxed{16/3}$$

3. Integrate $f(x, y) = x + 2y$ over the triangle pictured below.

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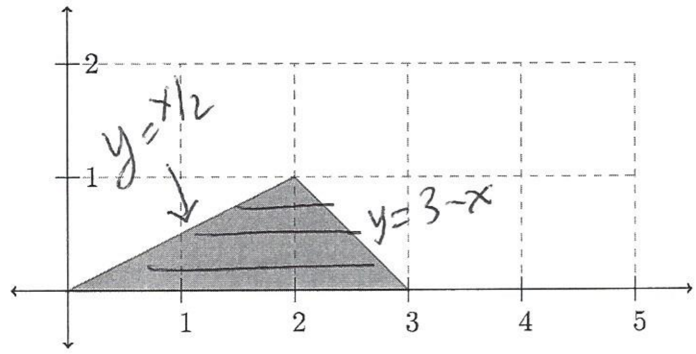

$0 \le y \le 1$

$2y \le x \le 3-y$

$$\int_0^1 \int_{2y}^{3-y} x + 2y \, dx \, dy$$

$$= \int_0^1 \frac{x^2}{2} + 2xy \Big|_{2y}^{3-y} dy$$

$$= \int_0^1 \frac{(3-y)^2}{2} + 2y(3-y) - \frac{(2y)^2}{2} - 2(2y)y \, dy$$

$$= \int_0^1 \frac{9}{2} - 3y + \frac{y^2}{\cancel{2}} + 6y - 2y^2 - 2y^2 - 4y^2 \, dy$$

$$= \int_0^1 \frac{9}{2} + 3y - \frac{15}{2} y^2 \, dy = \frac{9}{2} y + \frac{3}{2} y^2 - \frac{5}{2} y^3 \Big|_0^1$$

$$= \frac{9}{2} + \frac{3}{2} - \frac{5}{2} = \boxed{7/2}$$

4. Integrate $f(x, y, z) = x$ over the region in the first octant $[x \geq 0,\ y \geq 0,\ z \geq 0]$ bounded from above by the plane $x + 2y + z = 6$.

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$0 \leq z \leq 6 - x - 2y$ $\qquad$ $6 - x - 2y = 0$

$0 \leq x \leq 6 - 2y$ $\qquad$ $x = 6 - 2y$

$0 \leq y \leq 3$ $\qquad$ $6 - 2y = 0 \Rightarrow y = 3$

$$\int_0^3 \int_0^{6-2y} \int_0^{6-x-2y} x \, dz\, dx\, dy$$

$18 \cdot 9 = 81 \cdot 2 = 162$

$$27 \cdot 8 = 216 \qquad 36 \cdot 6 = 216$$

$$= \int_0^3 \int_0^{6-2y} xz \Big|_0^{6-x-2y} dx\, dy$$

$$\frac{64}{24} = \frac{\cancel{4} \cdot 6 \cdot 3 \cdot 3}{\cancel{4}} = 54$$

$$= \int_0^3 \int_0^{6-2y} 6x - x^2 - 2xy \, dx\, dy$$

$$= \int_0^3 3x^2 - \frac{x^3}{3} - x^2 y \Big|_0^{6-2y} dy$$

$$= \int_0^3 3(6-2y)^2 - \frac{(6-2y)^3}{3} - y(6-2y)^2$$

$$= \int_0^3 3(6-2y)^2 - \frac{(6-2y)^3}{3} - y(36 - 24y + 4y^2)\, dy$$

$$= \int_0^3 3(6-2y)^2 - \frac{(6-2y)^3}{3} - 36y + 24y^2 - 4y^3 \, dy$$

Using $u = 6 - 2y$, $du = -2y\, dy$

$$= \left[-\frac{(6-2y)^3}{2} + \frac{(6-2y)^4}{24} - 18y^2 + 8y^3 - y^4 \right] \Big|_0^3 \qquad \boxed{27}$$

$$= 0 + 0 - 18 \cdot 9 + 8 \cdot 27 - 81 + \frac{6^3}{2} - \frac{6^4}{24} = -\cancel{162} + \cancel{216} - 81 + 108 - \cancel{54}$$

5. Convert $(x, y, z) = (0, 3, 4)$ to both cylindrical and spherical coordinates.

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Cylindrical: $(3, \pi/2, 4)$

Spherical: $(5, \pi/2, \cos^{-1}(4/5))$.

6. Convert the following equations to spherical coordinates.

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(a) $z^2 = x^2 + y^2$.

(b) $z = x^2 + y^2$.

(c) $x^2 + y^2 + z^2 = 4$.

(a) $\cos^2\varphi = \sin^2\varphi \Rightarrow \boxed{\varphi = \pi/4.}$

(b) $\cos\varphi = \rho\sin^2\varphi \Rightarrow \boxed{\rho = \dfrac{\cos\varphi}{\sin^2\varphi}}$

(c) $\boxed{\rho = 2}$

Conversion Formulas

Cylindrical		Spherical	
$x = r\cos(\theta)$	$r = \sqrt{x^2+y^2}$	$x = \rho\cos(\theta)\sin(\phi)$	$\rho = \sqrt{x^2+y^2+z^2}$
$y = r\sin(\theta)$	$\tan(\theta) = \frac{y}{x}$	$y = \rho\sin(\theta)\sin(\phi)$	$\tan(\theta) = \frac{y}{x}$
$z = z$	$z = z$	$z = \rho\cos(\phi)$	$\cos(\phi) = \frac{z}{\rho}$