MATH 251: Quiz 3

June 18, 2015

Name: Solutions

1. Find the direction of steepest ascent for the function $f(x,y) = x^3 + 2xy + y^2$ at the point $(-1,1)$.

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Steepest Ascent $= \nabla f$

$$\nabla f = \langle 3x^2 + 2y,\ 2x + 2y \rangle$$

$$\nabla f\big|_{(-1,1)} = \langle 3+2,\ -2+2 \rangle = \boxed{\langle 5, 0 \rangle}$$

2. If $T(x,y,z) = x^2 + 3y - 2z^2$ and $\vec{c}(t) = \langle \sin(t), t, \cos(t) \rangle$, compute

$$\frac{d}{dt} T(\vec{c}(t)) \qquad \text{at } t = \frac{\pi}{4}.$$

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$$\nabla T = \langle 2x, 3, -4z \rangle$$

$c(t) = \langle \sin(t), t, \cos t \rangle$ $\qquad$ $c(\pi/4) = \langle \sqrt{2}/2, \pi/4, \sqrt{2}/2 \rangle$

$c'(t) = \langle \cos t, 1, -\sin(t) \rangle$ $\qquad$ $c'(\pi/4) = \langle \sqrt{2}/2, 1, -\sqrt{2}/2 \rangle$

$$\nabla T(c(t)) = \langle \sqrt{2}, 3, -2\sqrt{2} \rangle$$

$$\nabla T(\vec{c}(t)) \cdot \vec{c}'(t) = \langle \sqrt{2}, 3, -2\sqrt{2} \rangle \cdot \langle \sqrt{2}/2, 1, -\sqrt{2}/2 \rangle$$

$$= 1 + 3 + 2 = \boxed{6}$$

$x(1,1)=2$
$y(1,1)=2$
$z(1,1)=0$

3. Find $\frac{\partial f}{\partial t}$ at $s=1$, $t=1$, for $f(x,y,z)=2x^3+3y^2+5z$ if

$$x=s^2+t^2 \qquad y=s+t \qquad z=s^2-t^2$$

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$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}+\frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$=(6x^2)(2t)+6y(1)+5(-2t)$$

$$=6(2)^2\cdot 2\cdot 1+6\cdot 2\cdot 1+5(-2)(1)$$

$$=48+12-10=\boxed{50}$$

4. If z is given implicitly as a function of x and y by

$$x^2+2y^2-4z^2+3x-2y+z=8$$

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find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.

$F_x=2x+3$

$F_y=4y-2$

$F_z=-8z+1$

$$\boxed{\frac{\partial z}{\partial x}=-\frac{F_x}{F_z}=\frac{2x+3}{8z-1} \qquad \frac{\partial z}{\partial y}=-\frac{F_y}{F_z}=\frac{4y-2}{8z-1}}$$

5. Find and characterize the critical points of the function

$$f(x, y) = y^3 - 3xy - x^2 + 2x$$

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$f_x = 2 - 2x - 3y$

$f_y = 3y^2 - 3x = 0$

$\Rightarrow 3y^2 = 3x$

$x = y^2$

$0 = 2 - 2y^2 - 3y$

$2y^2 + 3y - 2 = 0$

$(2y - 1)(y + 2) = 0$

$$\boxed{\begin{array}{l} y = -2,\ x = 4 \\ y = 1/2,\ x = 1/4 \end{array}}$$

$f_{xx} = -2$

$f_{xy} = -3$

$f_{yy} = 6y$

$D = f_{xx} f_{yy} - f_{xy}^2$

$= -12y - 9$

$(4, -2) \to D = 24 - 9 > 0 \quad f_{xx} < 0 \Rightarrow$ maximum

$\cancel{(1/2, 1/4)}$ $(1/4, 1/2)$ $\to D = -6 - 9 < 0 \Rightarrow$ saddle point

$f \downarrow$

$3x+4y-\sqrt{11}\,z$ $g \downarrow$

6. Find the maximum value of ~~$3x+4y-z$~~ on the unit sphere $x^2+y^2+z^2=1$.

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$\nabla f = \langle 3, 4, -\sqrt{11} \rangle$ $\nabla g = \langle 2x, 2y, 2z \rangle$

$\underline{\nabla f = \lambda \nabla g}$

$3 = 2x\lambda \rightarrow x = \frac{3}{2\lambda}$

$4 = 2y\lambda \rightarrow y = 2/\lambda$

$-\sqrt{11} = 2z\lambda \rightarrow z = -\sqrt{11}/2\lambda$

$x^2+y^2+z^2=1$

$\left(\frac{3}{2\lambda}\right)^2 + \left(\frac{2}{\lambda}\right)^2 + \left(\frac{-\sqrt{11}}{2\lambda}\right)^2 = 1$

$\frac{9}{4\lambda^2} + \frac{4}{\lambda^2} + \frac{11}{4\lambda^2} = 1$

$9 + 16 + 11 = 4\lambda^2$

$36 = 4\lambda^2$

$9 = \lambda^3$

$\underline{\lambda = \pm 3}$

So

$x = \pm 1/2$

$y = \pm 2/3$

$z = \mp \sqrt{11}/6$

$\underline{\text{For } +}$: $3x+4y-\sqrt{11}\,z$

$= 3/2 + 8/3 + 11/6$

$= 9/6 + 16/6 + 11/6 = 36/6 = 6$

$\underline{\text{For } -}$ Same work $\rightarrow -6$

$\boxed{\text{Maximum value is } 6}$