MATH 251: Quiz 2

June 4, 2015

Name: Solutions

1. Find the equation of the plane through the point $(0, 2, 1)$ with normal vector $\langle 2, 3, 2\rangle$.

$$\vec{n} = \langle 2, 3, 2\rangle$$

$$d = \vec{n} \cdot \langle x_0, y_0, z_0\rangle = \langle 2,3,2\rangle \cdot \langle 0,2,1\rangle = 2\cdot 0 + 3\cdot 2 + 2\cdot 1$$
$$= 8$$

Plane: $\boxed{2x + 3y + 2z = 8}$

2. Find the equation of the plane through the points $(1, 2, 3)$, $(2, -1, 2)$ and $(-1, -1, -1)$.

$$\vec{v}_1 = (1,2,3) - (-1,-1,-1) = \langle 2,3,4\rangle$$

$$\vec{v}_2 = (2,-1,2) - (-1,-1,-1) = \langle 3,0,3\rangle$$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \langle 9-0, 12-6, 0-9\rangle$$
$$= \langle 9, 6, -9\rangle$$

$$d = \vec{n} \cdot \langle -1,-1,-1\rangle = -9 - 6 + 9 = -6$$

$\boxed{9x + 6y - 9z = -6}$ or $\boxed{3x + 2y - 3z = -2}$

3. For the vector-valued function $\vec{r}(t) = \langle 2e^t, t^3, \frac{1}{t}\rangle$, compute

(a) $\vec{r}'(t)$

(b) $\int_1^3 \vec{r}(t)\, dt$

a) $\vec{r}'(t) = \langle 2e^t, 3t^2, -1/t^2 \rangle$

b) $\int \vec{r}(t)\,dt = \langle 2e^t, t^4/4, \ln t \rangle + \vec{C}$

So $\int_1^3 \vec{r}(t)\,dt = \langle 2e^t, t^4/4, \ln t \rangle \Big|_1^3$

$= \langle 2(e^3 - e), \frac{81}{4} - \frac{1}{4}, \ln 3 - \ln 1 \rangle$

$= \boxed{\langle 2e(e^2-1), 20, \ln 3 \rangle}$

4. Find the tangent vector to the curve $\vec{r}(t) = \langle t^4, e^t + 2, 2t^2 + 3t + 1\rangle$ at the point $t = 2$.

Tangent vector $= \vec{r}'(2)$

$\vec{r}'(t) = \langle 4t^3, e^t, 4t + 3 \rangle$

$\vec{r}'(2) = \langle 4(2)^3, e^2, 8 + 3 \rangle$

$\boxed{= \langle 32, e^3, 11 \rangle}$

5. For the curve $\vec{r}(t) = \langle \cos(4t), 3t, \sin(4t) \rangle$, compute

(a) the length of $\vec{r}(t)$ between $t = 0$ and $t = 3$.

(b) the curvature of $\vec{r}(t)$ at $t = 1$.

(a) $\vec{r}\,'(t) = \langle -4\sin(4t), 3, 4\cos(4t) \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{16\sin^2(4t) + 9 + 16\cos^2(4t)}$

$= \sqrt{16 + 9} = 5$

So $\text{length} = \int_0^3 \|\vec{r}\,'(t)\|\,dt = \int_0^3 5\,dt = 5t\Big|_0^3 = \boxed{15}$

(b) Method 1: Arc length parametrization.

$s(t) = \int_0^t \|\vec{r}\,'(u)\|\,du = 5t$

So $s = t/5$.

Thus

$\vec{r}_1(s) = \langle \cos(4s/5), 3s/5, \sin(4s/5) \rangle$ is an arc length parametrization.

$\vec{r}_1\,'(s) = \langle -\frac{4}{5}\sin(4s/5), 3/5, +\frac{4}{5}\cos(4s/5) \rangle$

$\vec{r}_1\,''(s) = \langle -\frac{16}{25}\cos(4s/5), 0, -\frac{16}{25}\sin(4s/5) \rangle$

and $\kappa(s) = \|\vec{r}_1\,''(s)\| = \sqrt{\left(\frac{16}{25}\right)^2 \cos^2(4s/5) + \left(\frac{16}{25}\right)^2 \sin^2(4s/5)} = \frac{16}{25}$

Thus $\boxed{\kappa(1/5) = 16/25}$

Method 2: General Formu~~ll~~a

$$\vec{r}'(t) = \langle -4\sin(4t),\ 3,\ 4\cos(4t) \rangle$$

$$\vec{r}''(t) = \langle -16\cos(4t),\ 0,\ -16\sin(4t) \rangle$$

$$\vec{r}'(t) \times \vec{r}''(t) = \langle -48\sin(4t),\ \underbrace{-64\cos^2(4t) - 64\sin^2(4t)}_{=-64},\ 48\cos(4t) \rangle.$$

$$\|\vec{r}'(t) \times \vec{r}''(t)\| = \sqrt{(48)^2\sin^2(4t) + (64)^2 + (48)^2\cos^2(4t)}$$

$$= \sqrt{(48)^2 + (64)^2} = 8\sqrt{\underbrace{6^2+8^2}_{10^2}} = \underline{80}.$$

$$\|\vec{r}'(t)\|^3 = 5^3 = \underline{125}.$$

So $$\kappa(t) = \frac{\|\vec{r}'(t) \times \vec{r}''(t)\|}{\|\vec{r}'(t)\|^3} = \frac{80}{125} = \frac{16}{25}$$

and $$\boxed{\kappa(1) = \frac{16}{25}}$$