MATH 251: Practice 18

June 29, 2015

Name: Solution

Integrate $f(x, y, z) = z$ over the region above the cone $z^2 = x^2 + y^2$ $z \geq 0$ and inside the sphere of radius 2, $x^2 + y^2 + z^2 = 4$.

(a) Convert the boundary surfaces, $z^2 = x^2 + y^2$ and $x^2 + y^2 + z^2 = 4$, and $f(x, y, z) = z$ into spherical coordinates.

(b) Use the boundaries to set up bounds on ρ, θ, and ϕ.

(c) Evaluate the integral in spherical coordinates.

For reference, the integral in rectangular coordinates is

$$\int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} z \, dz \, dy \, dx$$

(a) $\varphi = \pi/4$, $\rho = 2$. $z = \rho \cos\varphi$.

(b) $0 \leq \rho \leq 2$, $0 \leq \varphi \leq \pi/4$, $0 \leq \theta \leq 2\pi$

(c) $$\int_0^{2\pi} \int_0^{\pi/4} \int_0^2 \rho \cos\varphi \cdot \rho^2 \sin\varphi \, d\rho \, d\varphi \, d\theta$$

$$= \int_0^{2\pi} \int_0^{\pi/4} \frac{\rho^4}{4} \cos\varphi \sin\varphi \Big|_0^2 \, d\varphi \, d\theta$$

$$= 4 \int_0^{2\pi} \int_0^{\pi/4} \cos\varphi \sin\varphi \, d\varphi \, d\theta$$

$u = \sin\varphi$

$du = \cos\varphi \, d\varphi$

$$= 4 \cdot \int_0^{2\pi} d\theta \cdot \frac{\sin^2\varphi}{2} \Big|_0^{\pi/4} = \int_0^{2\pi} d\theta = \boxed{2\pi}$$