Repeated Eigenvalues

Repeated eigenvalues can also cause difficulties when trying to find the general solution. These aren't too common in real-world scenarios, but they can happen and help to illustrate what happens as eigenvalues get closer and closer together.

Example. Analyze the system

$$\vec{x}\,' = \begin{bmatrix} 6 & -8 \\ 2 & -2 \end{bmatrix} \vec{x}.$$

$$A - \lambda I = \begin{bmatrix} 6-\lambda & -8 \\ 2 & -2-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (6-\lambda)(-2-\lambda) + 16$$
$$= \lambda^2 - 4\lambda - 12 + 16$$
$$= \lambda^2 - 4\lambda + 4 = (\lambda - 2)^2$$

Double Eigenvalue at 2.

$$A - 2I: \begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

Defective Eigenvalues

The main issue here is that we have a defective eigenvalue.

Power in the characteristic polynomial is strictly greater than the number of linearly independent eigenvectors.

→ Not enough information to get to a general solution.

$$\bar{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

- Need to find another solution somehow to get to the general solution.

Guessing a Second Solution

We've solved a similar issue like this before.

Repeated roots for second order equations.

$\rightarrow$ Multiplying by t.

$$\vec{x}' = A\vec{x} \qquad A = \begin{bmatrix} 6 & -8 \\ 2 & -2 \end{bmatrix}$$

Guess $\vec{x} = \vec{v}te^{2t}$

$$\vec{x}' = \underline{\vec{v}e^{2t}} + 2\vec{v}te^{2t} = A\vec{v}te^{2t}$$

New Guess: $\vec{x} = \vec{v}te^{2t} + \vec{w}e^{2t}$

$$\vec{x}' = \vec{v}e^{2t} + 2\vec{v}te^{2t} + 2\vec{w}e^{2t} = A\vec{v}te^{2t} + A\vec{w}e^{2t}$$

e^{2t}: $\vec{v} + 2\vec{w} = A\vec{w}$

te^{2t}: $2\vec{v} = A\vec{v}$ $\rightarrow$ Eigenvalue Equation

te^{2t}: $A\vec{v} - 2\vec{v} = \vec{0}$

$\vec{v}$ is an eigenvector of A with eigenvalue

$2 \rightarrow \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \vec{v}$

e^{2t}: $A\vec{w} - 2\vec{w} = \vec{v}$

$$\begin{bmatrix} 4 & -8 \\ 2 & -4 \end{bmatrix} \vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 & -8 & 2 \\ 2 & -4 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$2w_1 - 4w_2 = 1$ $\quad \vec{w} = \begin{bmatrix} 1/2 \\ 0 \end{bmatrix}$ works.

$$\boxed{\vec{x}_2(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} te^{2t} + \begin{bmatrix} 1/2 \\ 0 \end{bmatrix} e^{2t}}$$

$$\boxed{\vec{x}_1(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}}$$