Real and Distinct Eigenvalues

So, we know how we can find straight-line solutions to a linear, constant coefficient, homogeneous system, and these give us solutions to the system. Since we are dealing with a linear system, the next question we want to ask is if we can use this to give a general solution to the problem.

$\vec{v}e^{\lambda t}$ for λ an eigenvalue of A, and $\vec{v}$ a corresponding eigenvector.

$$\vec{x}' = A\vec{x}$$

A general solution should be

$$c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \dots + c_k\vec{x}_k(t)$$

So that we can meet every initial condition.

$$c_1\vec{v}_1e^{\lambda_1 t} + c_2\vec{v}_2e^{\lambda_2 t} \dots c_k\vec{v}_k e^{\lambda_k t}$$

at $t=0$

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k$$

So, if we want a general solution, that means we need to meet every initial condition. What does this mean?

For every vector $\vec{x}_0$, find coefficients $c_1, \dots, c_k$ so that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_k\vec{v}_k = \underbrace{\vec{x}_0}_{\text{vector in } \mathbb{R}^n}$$

$$\text{span}\{\vec{v}_1, \dots, \vec{v}_k\} = \mathbb{R}^n$$

Need: $\vec{v}_1, \dots, \vec{v}_k$ form a basis of $\mathbb{R}^n$

$k = n$ + linear independence.

We know this works if A has real and distinct eigenvalues.

→ Eigenvectors form a basis.

Theorem 0.1. If A has all real and distinct eigenvalues, then the general solution to

$$\vec{x}' = A\vec{x} \quad \text{is}$$

$$c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A and v_i is a corresponding eigenvector of A for λ_i.

→ Works for any size system; we will focus on 2-component mostly.

→ Abel's Thm.