Diagonalization

For the idea of what diagonalization is and why it might be helpful, let's look at an example.

Example. Find the general solution of the system

$$\vec{x}\,' = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \vec{x} + \begin{bmatrix} t \\ e^{-t} \end{bmatrix}.$$

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2x_1 + t \\ 3x_2 + e^{-t} \end{bmatrix}$$

$$x_1' = 2x_1 + t$$

$$x_1' - 2x_1 = t$$

$$\left(e^{-2t} x_1\right)' = t e^{-2t}$$

$$e^{-2t} x_1 = -\frac{1}{2} t e^{-2t} - \frac{1}{4} e^{-2t} + C$$

$$x_1 = -\frac{1}{2} t - \frac{1}{4} + C e^{2t}$$

$$x_2' = 3x_2 + e^{-t}$$

$$\left(e^{-3t} x_2\right)' = e^{-4t}$$

$$e^{-3t} x_2 = -\frac{1}{4} e^{-4t} + C$$

$$x_2 = -\frac{1}{4} e^{-t} + C e^{3t}$$

$$\boxed{\vec{x} = \begin{bmatrix} -\frac{1}{2} t - \frac{1}{4} \\ -\frac{1}{4} e^{-t} \end{bmatrix} + c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} e^{3t}}$$

So, if the matrix is diagonal, then the system is *decoupled*, and it becomes a collection of individual equations that we can solve. The question is, can we convert a system that is not diagonal into one that is?

Yes! We can do this with eigenvalues.

If A is an $n \times n$ matrix with n real distinct eigenvalues, then we can rewrite A as

$$A = PDP^{-1}$$ where

D is a diagonal matrix containing the eigenvalues of A.

P is a matrix whose columns are the eigenvectors of A in the same order as the eigenvalues are in D.

How does this solve the problem?

$$\vec{x}' = A\vec{x} + \vec{f}$$

Define $\vec{y} = P^{-1}\vec{x} \rightarrow \vec{x} = P\vec{y}$

$$(P\vec{y})' = A(P\vec{y}) + \vec{f}$$

$$P\vec{y}' = PD\underbrace{P^{-1}P}\vec{y} + \vec{f}$$

$$\vec{y}' = D\vec{y} + \underbrace{\boxed{P^{-1}f}}_{\vec{g}}$$

→ Solve this decoupled system.

$\vec{x} = P\vec{y}$ to convert the equation back.