Constant Coefficient Linear Systems

In order to use our linear algebra skills to approach systems of differential equations, we need to look at a particular type of system so make it work.

Two-Component Systems

$$\frac{dx}{dt} = F(x,y,t) \qquad \frac{dy}{dt} = G(x,y,t)$$

→ Linear Systems

→ Constant Coefficient Systems

$$\frac{dx}{dt} = ax + by + f_1(t)$$

$$\vec{x} = \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\frac{dy}{dt} = cx + dy + f_2(t)$$

$$\vec{x}\,' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \vec{x} + \underbrace{\begin{bmatrix} f_1(t) \\ f_2(t) \end{bmatrix}}_{=0 \;\Rightarrow\; \text{Homogeneous}}$$

Analyzing these Systems

There are two main ways we can interpret these systems.

$$\vec{x}' = A\vec{x}$$

→ Autonomous System

• Equilibrium Solutions

Want $\vec{x}' = 0$ → $A\vec{x} = 0$

Assume A is invertible ($\det(A) \neq 0$)

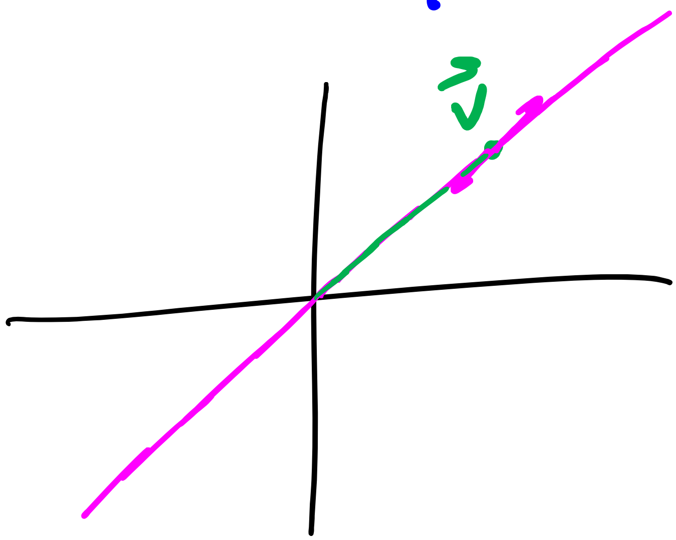
So only solution is $\boxed{\vec{x} = 0}$

→ Great, but not that much information.

Straight-Line Solutions

There wasn't too much information to be gained from equilibrium solutions. What else could we look at instead?

Find solutions that will go directly towards or away from the equilibrium solution at $\vec{0}$.



For the solution to go directly towards or away from the origin.

Need $\vec{v}\,' = \lambda \vec{v}$

$$\vec{x}\,' = \lambda \vec{x}$$
$$\shortparallel$$
$$A\vec{x}$$

To find these solutions, Need a vector $\vec{x}$ so that

$$\boxed{A\vec{x} = \lambda \vec{x}}$$

A second method

Another way to think about these solutions is to go back to what the solutions looked like for linear equations.

$$y' = ay \quad \rightarrow \quad y = ce^{at}$$

$$\vec{x}' = A\vec{x}$$

Plug in $\vec{x} = \vec{v}e^{\lambda t}$

$$\vec{x}' = \lambda \vec{v} e^{\lambda t}$$

$$\lambda \vec{v} e^{\lambda t} = A\vec{v}e^{\lambda t}$$

$$\boxed{A\vec{v} = \lambda \vec{v}}$$

Summary

If we have a system of the form

$$\vec{x}' = A\vec{x} \qquad A, \text{ constant matrix.}$$

then we can get a "straight-line" solution of the form

$$\vec{x}(t) = \vec{v} e^{\lambda t}$$

where λ is an eigenvalue of A with corresponding eigenvector $\vec{v}$.