Complex Eigenvalues

We know how to solve constant coefficient systems if the eigenvalues are real and distinct. What happens if they are complex-valued?

$\vec{x}' = A\vec{x}$

→ Eigenvalues and Eigen vectors.

Complex Eigenvalues?

Eigenvector will also be complex

Still get a complex-valued solution

$\vec{x}(t) = \vec{v}e^{\lambda t}$ $\quad$ $\vec{v}$ and λ both complex

→ Not helpful for modeling physical problems.

• Need to convert to real-valued solutions

$$e^{ix} = \cos(x) + i\sin(x)$$

The Return of Euler

As with second order equations, the trick we will use here is Euler's Formula, combined with

Theorem 0.1. *Assume that $\vec{x}(t)$ solves the system of differential equations*

$$\vec{x}\,'(t) = P(t)\vec{x}$$

for $P(t)$ a real-valued matrix function. If $\vec{x}(t)$ is complex-valued and can be decomposed into $\vec{x}(t) = \vec{u}(t) + i\vec{v}(t)$, for $\vec{u}$ and $\vec{v}$ real-valued, then $\vec{u}$ and $\vec{v}$ are also solutions to the same differential system.

Take $\vec{x}(t) = \vec{v}\, e^{\lambda t}$

$$= \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \left(e^{rt}\left(\cos(t) + i\sin(t)\right) \right)$$

→ Multiply everything through first.

→ Then split into real and imaginary parts.

Example. Find a real-valued general solution to the system

$$\vec{x}\,' = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} \vec{x}.$$

<u>Eigenvalues</u> $\quad A - \lambda I = \begin{bmatrix} 0-\lambda & -1 \\ 2 & 2-\lambda \end{bmatrix}$

$$\det(A-\lambda I) = (-\lambda)(2-\lambda) + 2$$

$$= \lambda^2 - 2\lambda + 2$$

$$\lambda = \frac{2 \pm \sqrt{4 - 4\cdot 2}}{2} = 1 \pm i$$

<u>Eigenvector</u>

$$A - (1+i)I = \begin{bmatrix} -1-i & -1 \\ 2 & 1-i \end{bmatrix}$$

$-1 - i + i - 1$

$$= \begin{bmatrix} (-1-i)(1-i) & -1(1-i) \\ 2 & 1-i \end{bmatrix}$$

$$= \begin{bmatrix} -2 & -(1-i) \\ 2 & 1-i \end{bmatrix} \vec{v} = \vec{0}$$

$$2v_1 + (1-i)v_2 = 0$$

$$v_1 = 1-i \qquad v_2 = -2$$

Eigenvalue: $1+i$ Eigenvector $\begin{bmatrix} 1-i \\ -2 \end{bmatrix}$

Solution $\begin{bmatrix} 1-i \\ -2 \end{bmatrix} e^{(1+i)t}$

$$\begin{bmatrix} 1-i \\ -2 \end{bmatrix}\left(e^{t}\left(\cos t + i \sin(t)\right)\right)$$

$$\begin{bmatrix} e^{t}(1-i)(\cos t + i \sin t) \\ -2e^{t}(\cos(t) + i \sin(t)) \end{bmatrix}$$

$$= \begin{bmatrix} e^{t}(\cos(t) - i\cos(t) + i\sin(t) + \sin(t)) \\ -2e^{t}(\cos(t) + i\sin(t)) \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} e^t\cos(t) + e^t\sin(t) \\ -2e^t\cos(t) \end{bmatrix}} + i\underbrace{\begin{bmatrix} e^t\sin(t) - e^t\cos(t) \\ -2e^t\sin(t) \end{bmatrix}}$$

General Solution

$$\boxed{c_1\begin{bmatrix} e^t\cos(t) + e^t\sin(t) \\ -2e^t\cos(t) \end{bmatrix} + c_2\begin{bmatrix} e^t\sin(t) - e^t\cos(t) \\ -2e^t\sin(t) \end{bmatrix}}$$