Complex Eigenvalues

We know how to solve constant coefficient systems if the eigenvalues are real and distinct. What happens if they are complex-valued?

The Return of Euler

As with second order equations, the trick we will use here is Euler's Formula, combined with

Theorem 0.1. *Assume that $\vec{x}(t)$ solves the system of differential equations*

$$\vec{x}\,'(t) = P(t)\vec{x}$$

for $P(t)$ a real-valued matrix function. If $\vec{x}(t)$ is complex-valued and can be decomposed into $\vec{x}(t) = \vec{u}(t) + i\vec{v}(t)$, for $\vec{u}$ and $\vec{v}$ real-valued, then $\vec{u}$ and $\vec{v}$ are also solutions to the same differential system.

Example. Find a real-valued general solution to the system

$$\vec{x}\,' = \begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix} \vec{x}.$$