Variation of Parameters

A second method that can be used to solve non-homogeneous equations is variation of parameters. This is a method that works in general for all second-order linear equations, but does not always result in answers that can be found explicitly.

Equation $y'' + p(t)y' + q(t)y = g(t)$

Assume we know $y_1(t)$ and $y_2(t)$ solve

$$y'' + p(t)y' + q(t)y = 0$$

So $C_1 y_1(t) + C_2 y_2(t)$ also solves

<u>Goal</u> Try $y_p(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$ and figure out what u_1, u_2 need to be to solve the non-homogeneous problem.

Assume that we have the equation

$$y'' + p(t)y' + q(t)y = g(t)$$

and we know that the general solution to the homogeneous equation is $C_1y_1(t)+C_2y_2(t)$. To solve the non-homogeneous problem, we set $y = u_1y_1 + u_2y_2$ and plug it in.

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'$$

$$= u_1y_1' + u_2y_2' + \underbrace{u_1'y_1 + u_2'y_2}_{=0}$$

Choose to make this happen.

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''$$

$$y'' + p(t)y' + q(t)y = g(t)$$

$$\left[u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''\right] + p(t)\left[u_1y_1' + u_2y_2'\right] + q(t)\left(u_1y_1 + u_2y_2\right) = g(t)$$

$$u_1'y_1' + u_2'y_2' + u_1\underbrace{\left[y_1'' + p(t)y_1' + q(t)y_1\right]}_{0}$$

$$+ u_2\underbrace{\left[y_2'' + p(t)y_2' + q(t)y_2\right]}_{0} = g(t)$$

$$\begin{cases} u_1'y_1' + u_2'y_2' = g(t) \\ u_1'y_1 + u_2'y_2 = 0 \end{cases}$$

Two equations for the two unknown functions u_1' and u_2'.

The solution

From here, we can figure out what u_1 and u_2 should be.

$$(u_1' y_1' + u_2' y_2' = g(t))\, y_1$$
$$-\quad (u_1' y_1 + u_2' y_2 = 0)\, y_1'$$

$$u_2' y_2' y_1 - u_2' y_2 y_1' = g(t) y_1$$
$$u_2' (y_1 y_2' - y_2 y_1') = g(t) y_1$$

$$\boxed{u_2' = \frac{g(t) y_1}{y_1 y_2' - y_1' y_2}}$$

Not zero because Fundamental set!

$$\boxed{u_1' = -\frac{g(t) y_2}{y_1 y_2' - y_1' y_2}}$$

Solution

$$y_p(t) = u_1 y_1 + u_2 y_2$$

$$\boxed{y_p(t) = -y_1(t)\int_{t_0}^{t}\frac{g(s)\,y_2(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}\,ds + y_2(t)\int_{t_0}^{t}\frac{g(s)\,y_1(s)}{y_1(s)y_2'(s) - y_1'(s)y_2(s)}\,ds}$$

General Solution

$$\boxed{y(t) = y_p(t) + C_1 y_1(t) + C_2 y_2(t)}$$