# Undetermined Coefficients

Based on the results before, we know that the general solution to a non-homogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

is

$y_c(t)$

$$y_p(t) + C_1 y_1(t) + C_2 y_2(t)$$

where $y_p$ is any solution of the non-homogeneous problem and $y_1$ and $y_2$ are the linearly independent solutions of the corresponding homogeneous problem. In the case of constant-coefficient equations, we know how to get $y_1$ and $y_2$. Then we just need to find $y_p$.

We can find this $y_p$ however we want.

2 Methods

- → Undetermined Coefficients
- → Variation of Parameters.

The first method we have for finding this solution is the "Method of Undetermined Coefficients." Let's look at this method through an example.

**Example.** Find the general solution to the differential equation

$$y'' + 3y' - 4y = 3e^{2t}$$

Homogeneous Problem

$r^2 + 3r - 4 = 0$

$(r+4)(r-1) = 0$

Gen Sol: $\boxed{c_1 e^{-4t} + c_2 e^{t}}$

To find $y_p(t)$:

Since derivatives of $e^{2t}$ still involve $e^{2t}$ something times $e^{2t}$ might be a good candidate for $y_p(t)$.

A is the Undetermined Coefficient

Try $y_p(t) = A e^{2t}$

$y_p' = 2Ae^{2t}$

$y_p'' = 4Ae^{2t}$

$$y_p'' + 3y_p' - 4y_p = 4Ae^{2t} + 3(2Ae^{2t}) - 4(Ae^{2t})$$

$$= 4Ae^{2t} + 6Ae^{2t} - 4Ae^{2t}$$

$$= 6Ae^{2t}$$

To solve Non-homogeneous problem, want this to be $3e^{2t}$

Choose $A = 1/2$ so that

$$y_p(t) = \frac{1}{2}e^{2t}$$ ← Satisfies Non-homogeneous problem

General Solution for Non-homogeneous Prob

$$\boxed{y(t) = \frac{1}{2}e^{2t} + C_1 e^{-4t} + C_2 e^{t}}$$

The main point here that helped us solve this problem was that exponential functions repeat themselves when we take derivatives, and that we had a constant coefficient equation. What other functions do this?

$$(e^{ax})' = a\underbrace{e^{ax}}$$

Polynomials

$$(At+B)' = A$$

Sine + Cosine

$$(\sin(2t))' = 2\cos(2t)$$

$$(\sin(2t))'' = -4\sin(2t)$$