Repeated Roots Issues

Consider the second-order linear differential equation

$$y'' - 2y' + y = 0$$

What does our normal method get for a general solution?

Characteristic Equation $\quad r^2 - 2r + 1 = 0$

$$(r-1)^2 = 0$$

Only get one value of r.

"General solution" $\quad c_1 e^t + c_2 e^t$

$$= (c_1 + c_2) e^t$$

→ Really only one function.

$$y_1 y_2' - y_1' y_2 = 0$$

• This is not enough!

Reduction of Order

We need a way to take one solution of a second-order linear equation and use this to get another. One method of doing this, and the way presented here, is reduction of order.

Know $C\underbrace{e^t}$ is a solution for any C.

Try $y(t) = v(t)\underbrace{e^t}$ to hunt for this solution

→ plug into the original differential equation

→ Get a differential equation for $v(t)$.

Hopefully, e^t being a solution will make the equation for $v(t)$ simpler.

Example. Find the general solution to

$$y'' - 2y' + y = 0$$

Know e^t is one solution.

Plug in $y_2 = v(t)e^t$

$$y_2' = v'(t)e^t + v(t)e^t$$

$$y_2'' = v''(t)e^t + v'(t)e^t + v'(t)e^t + v(t)e^t$$

$$= v''(t)e^t + 2v'(t)e^t + v(t)e^t$$

$$y'' - 2y' + y = [v''(t)e^t + 2v'(t)e^t + v(t)e^t]$$

$$-2[v'(t)e^t + v(t)e^t] + [v(t)e^t]$$

$$= v''(t)e^t + v'(t)\cancel{[2e^t - 2e^t]} + v(t)\cancel{[e^t - 2e^t + e^t]}$$

$$v''(t)e^t = 0 \to \boxed{v''(t) = 0}$$

$$v(t) = At + B$$

$$y_2(t) = (At + B)e^t \to \boxed{y_2(t) = te^t}$$