General Reduction of Order

The idea of Reduction of Order works for any second (or higher) order linear differential equation. Assume we have one solution $y_1(t)$ to the differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

To use reduction of order, we search for a solution $y_2(t)$ as $v(t)y_1(t)$.

$$y_2(t) = v(t)y_1(t)$$

$$y_2'(t) = v'(t)y_1(t) + v(t)y_1'(t)$$

$$y_2''(t) = v''(t)y_1(t) + v'(t)y_1'(t) + v'(t)y_1'(t) + v(t)y_1''(t)$$

$$= v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t)$$

$$y'' + p(t)y' + q(t)y = [v''y_1 + 2v'y_1' + vy_1'']$$

$$+ p(t)[v'y_1 + vy_1'] + q(t)[vy_1]$$

$$= v''[y_1] + v'[2y_1' + p(t)y_1] + v\boxed{\cancel{[y_1'' + p(t)y_1' + q(t)y_1]}} = 0$$

0

$$v'' y_1 + \left[2y_1' + p(t)y_1\right] v' = 0$$

Let $u = v'$

$$u' y_1 + \left[2y_1' + p(t)y_1\right] u = 0$$

→ First Order Linear Equation!

→ Solvable by Integrating factors.

→ Find u

→ Integrate one more time to get v.

→ Multiply by y_1 to get the solution $\boxed{y_2}$.

Example. Consider the differential equation

$$t^2y'' + 2ty' - 6y = 0$$

(a) Verify that $y_1(t) = t^2$ is a solution to this differential equation.

(b) Use reduction of order to find a second solution.

(a) $\underline{t^2(2)} + \underline{2t(2t)} - \underline{6t}^2 = 0$ ✓

(b) $y_2(t) = v(t)\,t^2$

$y_2' = v' \cdot t^2 + 2t\,v$

$y_2'' = v'' \cdot t^2 + 2tv' + 2tv' + 2v$

$t^2y'' + 2ty' - 6y = t^2\left[v'' \cdot t^2 + 4tv' + 2v\right]$

$+ 2t\left[v't^2 + 2tv\right] - 6\left[vt^2\right]$

$= v''\left[t^4\right] + v'\left[4t^3 + 2t^3\right] + v\left[\cancel{2t^2 + 4t^2 - 6t^2}\right]$

$v'' \cdot t^4 + v' \cdot 6t^3 = 0$

$u = v'$

$$u' \cdot t^4 + u \cdot 6t^3 = 0$$

$$u' + \frac{6}{t} u = 0$$

$$\mu(t) = e^{\int \frac{6}{t} dt} = e^{6 \ln|t|} = t^6$$

$$u' \cdot t^6 + 6t^5 u = 0$$

$$(t^6 u)' = 0$$

$$t^6 u = C \rightarrow u = \frac{C}{t^6}$$

$$v' = u = \frac{C}{t^6} \rightarrow v = \frac{C}{-5} t^{-5} + D$$

$$y_2 = v \cdot t^2 = (At^{-5} + D)t^2$$

$$= A\boxed{t^{-3}} + Dt^2$$

Check: $y_2(t) = t^{-3}$ also solves the equation.