Non-Homogeneous Linear Equations

Now, we want to look at non-homogeneous equations and think about how we would solve them. The most general form of a second-order, linear, non-homogeneous equation is

$$y'' + p(t)y' + q(t)y = g(t)$$

for functions $p(t)$, $q(t)$, and $g(t)$.

$g(t) \neq 0$

How does $g(t) \neq 0$ change what the solutions look like?

For Homogeneous Eqns:

$c_1 y_1(t) + c_2 y_2(t)$

How can we try to solve these equations? At least for the constant-coefficient versions, we know how to solve the corresponding homogeneous equation. Can we get back to that equation somehow?

$$ay'' + by' + cy = \cancel{0}\ g(t)$$

Assume we have y_1 and y_2 that solve

$$y'' + p(t)y' + q(t)y = g(t)$$

Know

$$y_1'' + p(t)y_1' + q(t)y_1 = g(t)$$

$$y_2'' + p(t)y_2' + q(t)y_2 = g(t)$$

$$\underbrace{y_1'' - y_2''} + p(t)[y_1' - y_2'] + q(t)[y_1 - y_2] = 0$$

$$(y_1 - y_2)'' + p(t)(y_1 - y_2)' + q(t)(y_1 - y_2) = 0$$

$u = y_1 - y_2 \Rightarrow$

$$\boxed{u'' + p(t)u' + q(t)u = 0}$$

This work results in the following theorem:

Theorem 0.1. *Let $y_1(t)$ and $y_2(t)$ be two solutions to the non-homogeneous equation*

$$y'' + p(t)y' + q(t)y = g(t).$$

Then, the difference $y_1 - y_2$ solves the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

What does this do for us?
- We know how to solve the homog. version.

$$y_1 = \boxed{y_2} + y_c$$

Solves the homogeneous eqn.
"Complementary sol'n"

- Can get from any solution to non-homog. to any other by adding a solution to the homogeneous equation.
- We just need 1 solution to the non-homog. problem.

"particular soln"

$$\boxed{y_p + C_1 y_1(t) + C_2 y_2(t)}$$